Secondary Mathematics III: An Integrated Approach
Module 2
Logarithmic Functions

By

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In partnership with the Utah State Office of Education

Secondary Mathematics III
Module 2 – Logarithmic Functions

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2.1 Log Logic
A Develop Understanding Task

We began thinking about logarithms as inverse functions for exponentials in Tracking the Tortoise. Logarithmic functions are interesting and useful on their own. In the next few tasks, we will be working on understanding logarithmic expressions, logarithmic functions, and logarithmic operations on equations.

We showed the inverse relationship between exponential and logarithmic functions using a diagram like the one below:

Input		Output

$$x = 3 \xrightarrow{f(x) = 2^x} 2^3 = 8 \xrightarrow{\boxed{f^{-1}(x) = \log_2 x}} 3$$

We could summarize this relationship by saying:

$2^3 = 8$ so, $\log_2 8 = 3$

Logarithms can be defined for any base used for an exponential function. Base 10 is popular. Using base 10, you can write statements like these:

$10^1 = 10$ so, $\log_{10} 10 = 1$

$10^2 = 100$ so, $\log_{10} 100 = 2$

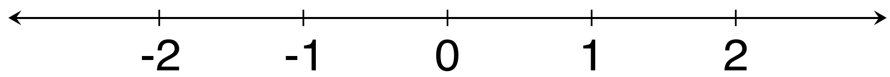
$10^3 = 1000$ so, $\log_{10} 1000 = 3$

The notation is a little strange, but you can see the inverse pattern of switching the inputs and outputs.

The next few problems will give you an opportunity to practice thinking about this pattern and possibly make a few conjectures about other patterns that you may notice with logarithms.

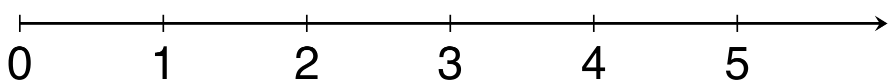
Place the following expressions on the number line. Use the space below the number line to explain how you knew where to place each expression.

1. A. $\log_3 3$ B. $\log_3 9$ C. $\log_3 \frac{1}{3}$ D. $\log_3 1$ E. $\log_3 \frac{1}{9}$



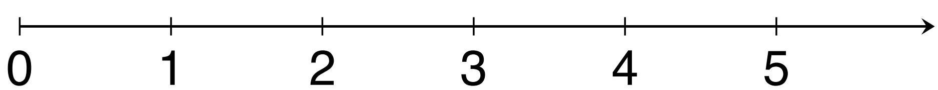
Explain: ____

2. A. $\log_3 81$ B. $\log_{10} 100$ C. $\log_8 8$ D. $\log_5 25$ E. $\log_2 32$



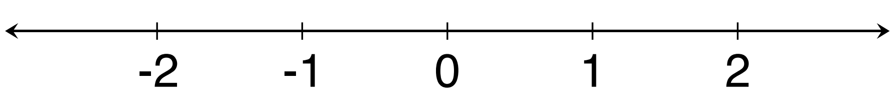
Explain: ____

3. A. $\log_7 7$ B. $\log_9 9$ C. $\log_{11} 1$ D. $\log_{10} 1$



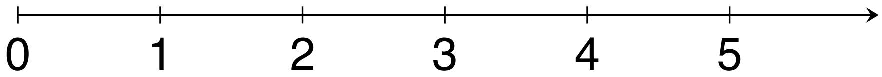
Explain: ____

4. A. $\log_2 \left(\frac{1}{4}\right)$ B. $\log_{10} \left(\frac{1}{1000}\right)$ C. $\log_5 \left(\frac{1}{125}\right)$ D. $\log_6 \left(\frac{1}{6}\right)$



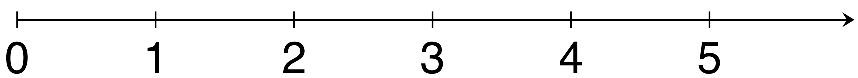
Explain: ____

5. A. $\log_4 16$ B. $\log_2 16$ C. $\log_8 16$ D. $\log_{16} 16$



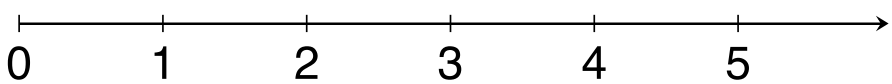
Explain: ____

6. A. $\log_2 5$ B. $\log_5 10$ C. $\log_6 1$ D. $\log_5 5$ E. $\log_{10} 5$



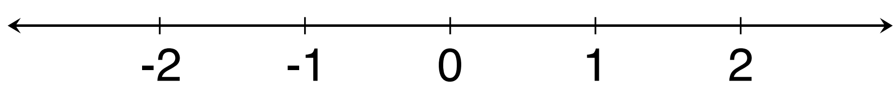
Explain: ____

7. A. $\log_{10} 50$ B. $\log_{10} 150$ C. $\log_{10} 1000$ D. $\log_{10} 500$



Explain: ____

8. A. $\log_3 3^2$ B. $\log_5 5^{-2}$ C. $\log_6 6^0$ D. $\log_4 4^{-1}$ E. $\log_2 2^3$



Explain: ____

Based on your work with logarithmic expressions, determine whether each of these statements is always true, sometimes true, or never true. If the statement is sometimes true, describe the conditions that make it true. Explain your answers.

9. The value of $\log_b x$ is positive.

Explain: ____

10. $\log_b x$ is not a valid expression if x is a negative number.

Explain: ____

11. $\log_b 1 = 0$ for any base, b > 1.

Explain: ____

12. $\log_b b = 1$ for any b > 1.

Explain: ____

13. $\log_2 x < \log_3 x$ for any value of x.

Explain: ____

14. $\log_{\mathrm{b}} b^n = n$ for any b > 1.

Explain: ____

Log Logic – Teacher Notes

A Develop Understanding Task

Purpose: The purpose of this task is to develop students understanding of logarithmic expressions and to make sense of the notation. In addition to evaluating log expressions, student will compare expressions that they cannot evaluate explicitly. They will also use patterns that they have seen in the task and the definition of a logarithm to justify some properties of logarithms.

Core Standards Focus:

F.BF.5 (+) Understand the inverse relationship between exponents and logarithms and use this relationship to solve problems involving logarithms and exponents.

F.LE.4 For exponential models, express as a logarithm the solution to $ab^{ct} = d$ where a, c, and d are numbers and the base b is 2, 10, or e; evaluate the logarithm using technology.

Note for F.LE.4: *Consider extending this unit to include the relationship between properties of logarithms and properties of exponents, such as the connection between the properties of exponents and the basic logarithm property that log xy = log x + log y.*

Related Standards: F.BF.4

Launch (Whole Class): Begin the task by working through each of the examples on page 1 of the task with students. Tell them that since we know that logarithmic functions and exponential functions are inverses, the definition of a logarithm is:

If $b^x = n$ then $\log_b n = x$ for $b > 1$

Keep this relationship posted where students can refer to it during their work on the task.

Explore (Small Group): The task begins with expressions that will generate integer values. In the beginning, encourage students to use the pattern expressed in the definition to help find the values. If they don't know the powers of the base numbers, they may need to use calculators to identify them. For instance, if they are asked to evaluate $\log_2 32$, they may need to use the calculator to find that $2^5 = 32$. (Author's note: I hope this wouldn't be the case, but the emphasis in this task is on reasoning, not on arithmetic skill.) Thinking about these values will help to review integer exponents.

Starting at #5, there are expressions that can only be estimated and placed on the number line in a reasonable location. Don't give students a way to use the calculator to evaluate these expressions directly; again the emphasis is on reasoning and comparing.

As you monitor students as they work, keep track of students that have interesting justifications for their answers on problems #9 – 15 so that they can be included in the class discussion.

Discuss (Whole Class): Begin the discussion with #2. For each log expression, write the equivalent exponential equation like so:

$\log_3 81 = 4 \quad 3^4 = 81$

This will give students practice in seeing the relationship between exponential functions and logarithmic functions. Place each of the values on the number line.

Move the discussion to #4 and proceed in the same way, giving students a brush-up on negative exponents.

Next, work with question #5. Since students can't calculate these expressions directly, they will have to use logic to figure this out. One strategy is to first put the expressions in order from smallest to biggest based on the idea that the bigger the base, the smaller the exponent will need to be to get 16. (Be sure this idea is generalized by the end of the discussion of #5.) Once the numbers are in order, then the approximate values can be considered based upon known values for a particular base.

Work on #7 next. In this problem, the bases are the same, but the arguments are different. The expressions can be ordered based on the idea that for a given base, $b > 1$, the greater the argument, the greater the exponent will need to be.

Finally, work through each of problems 9 – 15. This is an opportunity to develop a number of the properties of logarithms from the definitions. After students have justified each of the properties that are always true (#10, 11, 12, and 14), these should be posted in the classroom as agreed-upon properties that can be used in future work.

Aligned Ready, Set, Go: ***Logarithmic Functions 2.1***

Ready, Set, Go!

Ready

Topic: Graphing exponential equations

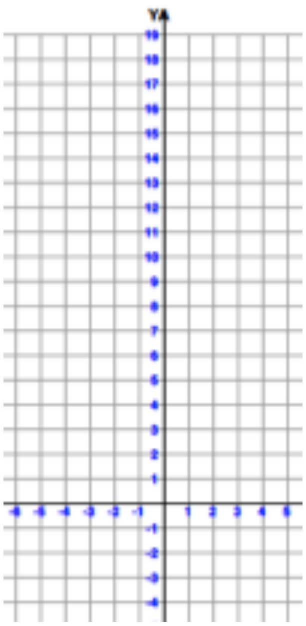
Graph each function over the domain $\{-4 \leq x \leq 4\}$.

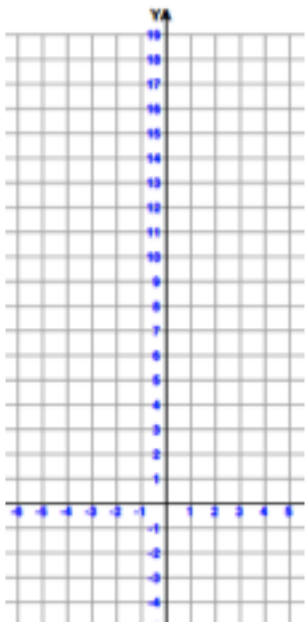
1. $y = 2^x$

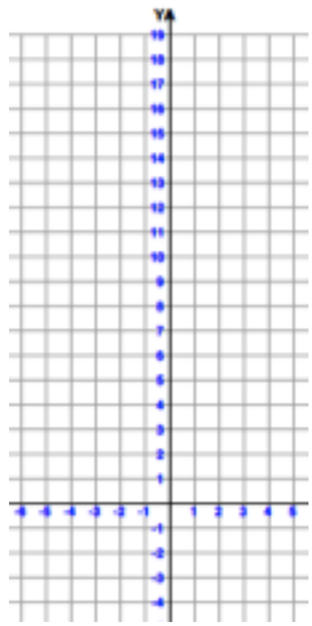
2. $y = 2 \cdot 2^x$

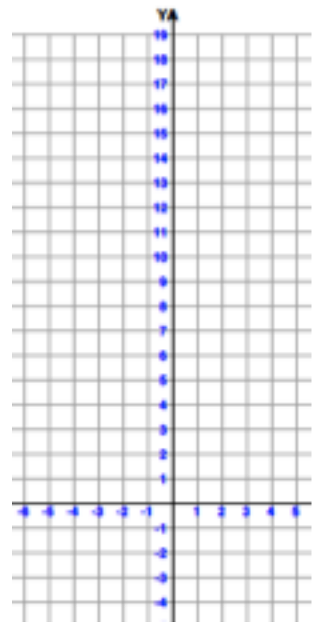
3. $y = \left(\frac{1}{2}\right)^x$

4. $y = 2\left(\frac{1}{2}\right)^x$









5. Compare graph #1 to graph #2. Multiplying by 2 should generate a dilation of the graph, but the graph looks like it has been translated vertically. How do you explain that?

6. Compare graph #3 to graph #4. Is your explanation in #5 still valid for these two graphs? Explain.

Name Logarithmic Functions | 2.1

Set

Topic: Evaluating logarithmic functions

Arrange the following expressions in numerical order from smallest to largest. Do not use a calculator. Be prepared to explain your logic.

	A	B	C	D	E
7.	$\log_2 32$	$\log_7 343$	$\log_{35} 1$	$\log_{15} 225$	$\log_{11} 11$
8.	$\log_3 81$	$\log_5 125$	$\log_8 8$	$\log_4 1$	$\log 100$
9.	$\log_7 45$	$\log_3 12$	$\log_4 12$	$\log_3 30$	$\log_x x$
10.	$\log_x \frac{1}{x^2}$	$\log_5 \frac{1}{5}$	$\log_2 \frac{1}{8}$	$\log \frac{1}{10,000}$	$\log_x 1$
11.	$\log 200$	$\log 0.02$	$\log_2 10$	$\log_2 \frac{1}{10}$	$\log_2 200$

Answer the following questions. If yes, give an example or the answer. If no, explain why not.

12. Is it possible for a logarithm to equal a negative number?

13. Is it possible for a logarithm to equal zero?

14. Does $log_x 0$ have an answer?

15. Does $log_x 1$ have an answer?

16. Does $log_x x^5$ have an answer?

Go

Topic: Properties of Exponents

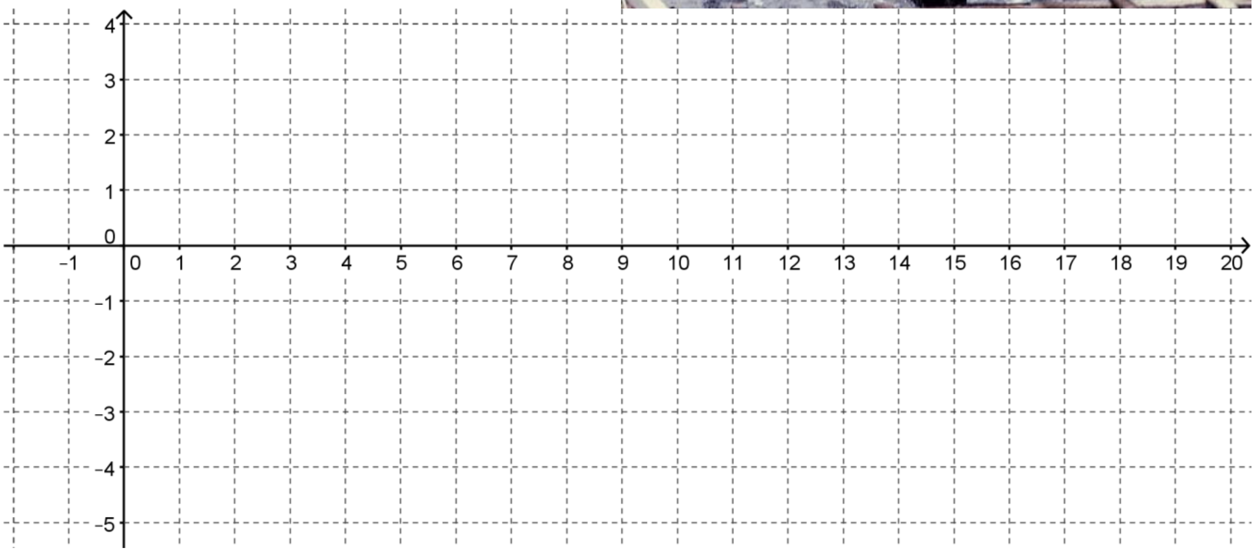
Write each expression as an integer or a simple fraction.

17. 27^0
18. $11(-6)^0$
19. -3^{-2}
20. 4^{-3}
21. $\frac{9}{2^{-1}}$
22. $\frac{4^3}{8^0}$
23. $\frac{4^0}{2^{-5}}$
24. $3\left(\frac{29^3}{11^5}\right)^0$
25. $42 \cdot 6^{-4}$
26. $\frac{3}{6^{-1}}$
27. $\frac{7^{-2}}{4^{-1}}$
28. $\frac{32^{-1}}{4^{-1}}$

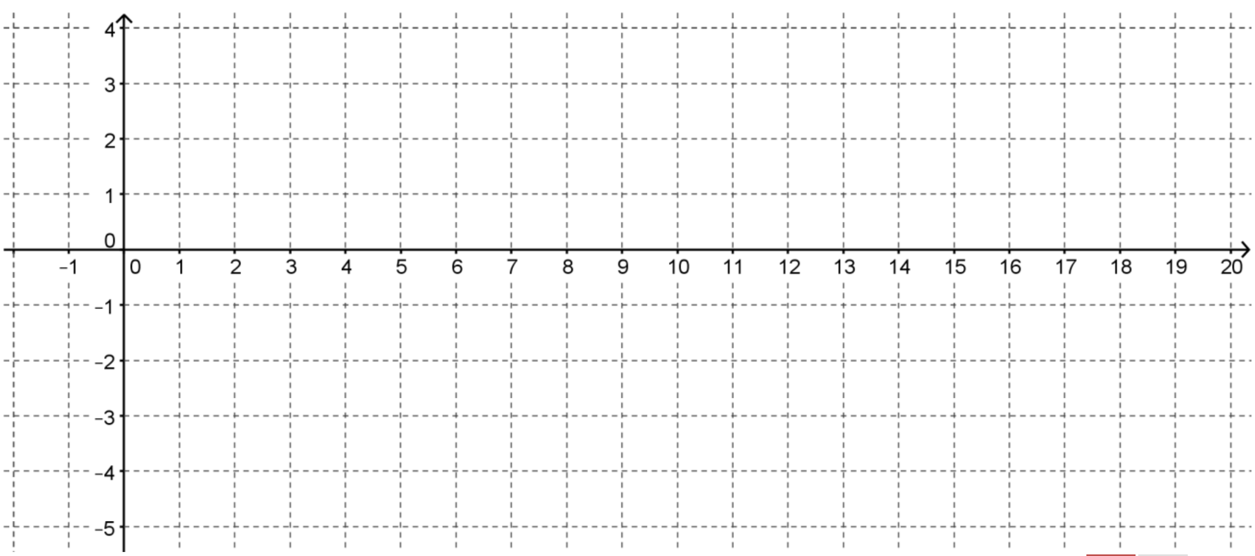
2.2 Falling Off A Log
A Solidify Understanding Task

1. Construct a table of values and a graph for each of the following functions. Be sure to select at least two values in the interval 0 < x < 1.

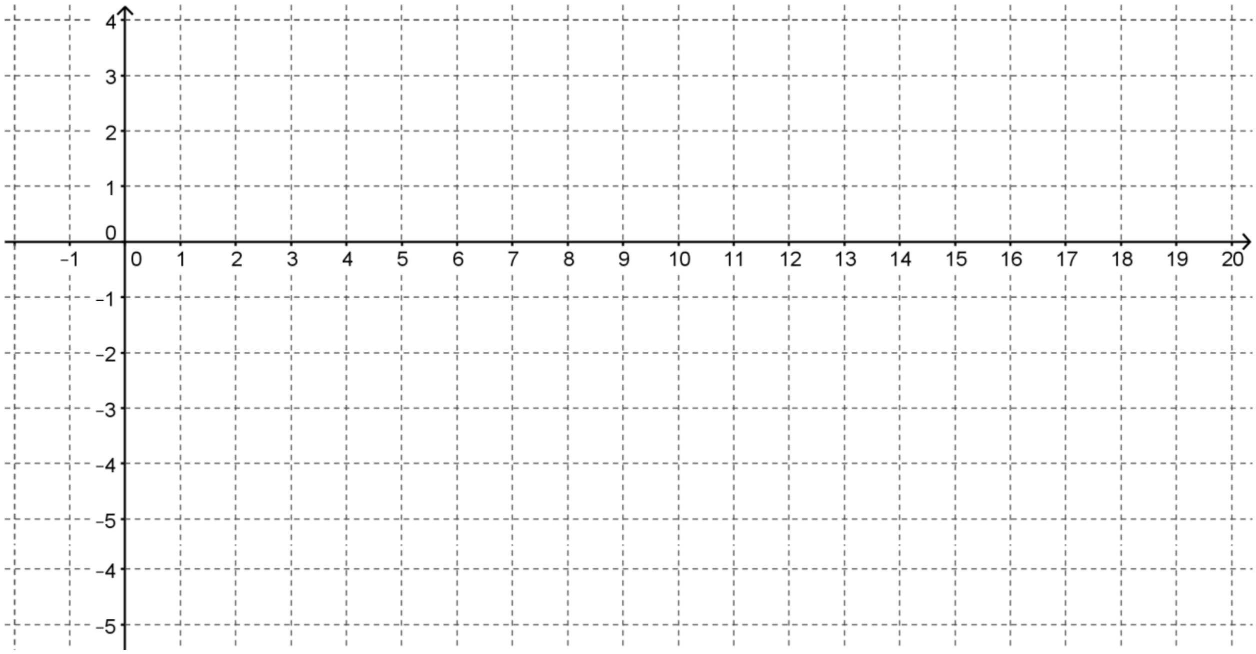
a) $f(x) = \log_2 x$



b) $g(x) = \log_3 x$



c) $h(x) = \log_4 x$



d) $k(x) = \log_{10} x$

2. How did you decide what values to use for *x* in your table?

3. How did you use the *x* values to find the y values in the table?

4. What similarities do you see in the graphs?

5. What differences do you observe in the graphs?

6. What is the effect of changing the base on the graph of a logarithmic function?

a) Let's focus now on $k(x) = \log_{10} x$ so that we can use technology to observe the effects of changing parameters on the function. Because base 10 is a very commonly used base for exponential and logarithmic functions, it is often abbreviated and written without the base, like this: $k(x) = \log x$.

b) Use technology to graph $y = \log x$. How does the graph compare to the graph that you constructed?

e) How do you predict that the graph of $y = a + \log x$ will be different from the graph of $y = \log x$?

f) Test your prediction by graphing $y = a + \log x$ for various values of a. What is the effect of a? Make a general argument for why this would be true for all logarithmic functions.

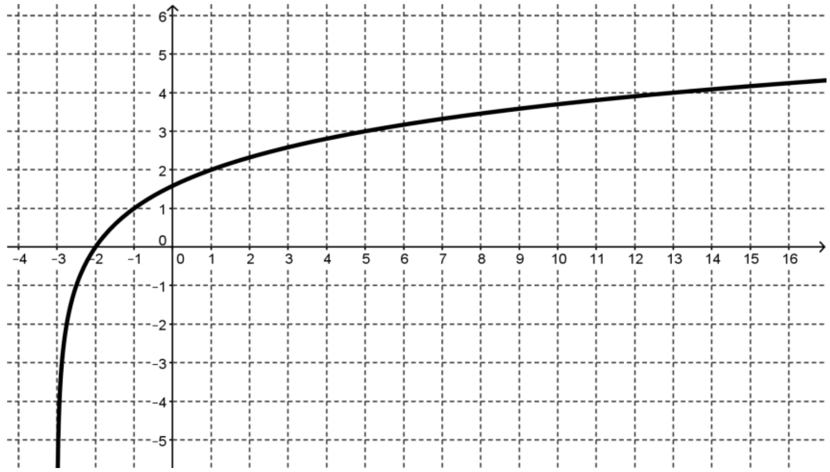
g) How do you predict that the graph of $y = \log(x + b)$ will be different from the graph of $y = \log x$?

h) Test your prediction by graphing $y = \log(x + b)$ for various values of b.

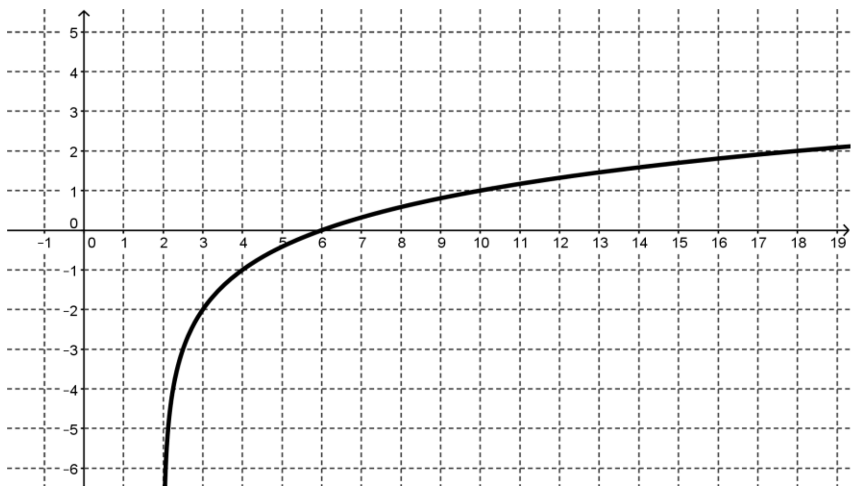
- What is the effect of adding b?
- What will be the effect of subtracting b?
- Make a general argument for why this is true for all logarithmic functions.

7. Write an equation for each of the following functions that are transformations of $f(x) = \log_2 x$.

a.

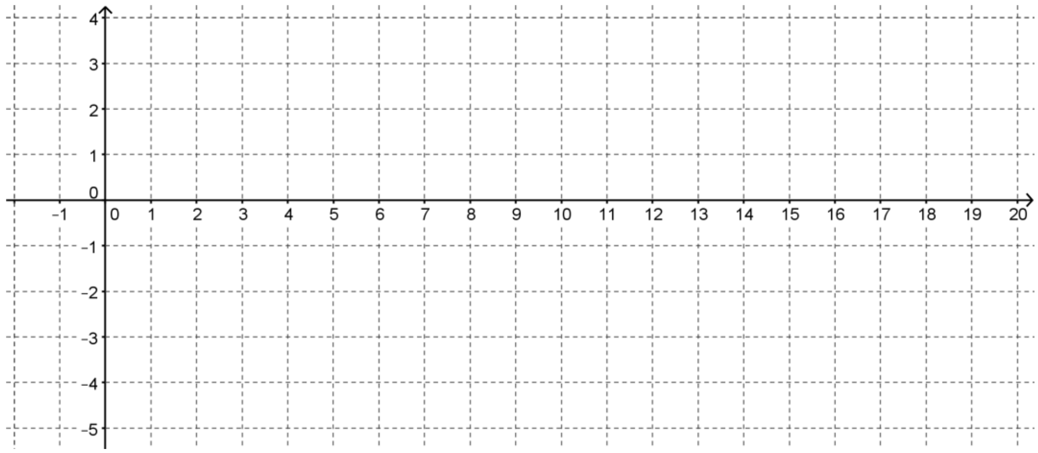


b.

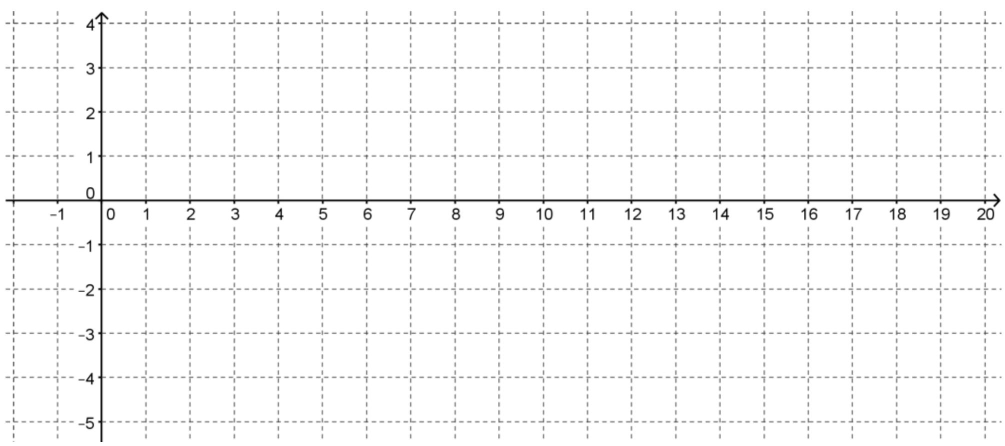


8. Graph and label each of the following functions:

a. $f(x) = 2 + \log_2(x-1)$



b. $g(x) = -1 + \log_2(x+2)$



9. Compare the transformation of the graphs of logarithmic functions with the transformation of the graphs of quadratic functions.

Falling Off A Log – Teacher Notes
A Solidify Understanding Task

Note to Teachers: Access to graphing technology is necessary for this task.

Purpose: The purpose of this task is to build on students' understanding of a logarithmic function as the inverse of an exponential function and their previous work in determining values for logarithmic expressions to find the graphs of logarithmic functions of various bases. Students use technology to explore transformations with log graphs in base 10 and then generalize the transformations to other bases.

Core Standards Focus:

F.BF.5 (+) Understand the inverse relationship between exponents and logarithms and use this relationship to solve problems involving logarithms and exponents.

Note for F. BF: Use transformations of functions to find more optimum models as students consider increasingly more complex situations.

For F.BF.3, note the effect of multiple transformations on a single function and the common effect of each transformation across function types. Include functions defined only by a graph.

Extend F.BF.4a to simple rational, simple radical, and simple exponential functions; connect F.BF.4a to F.LE.4.

Related Standards: F.LE.4

Launch (Whole Class): Begin class by reminding students of the work they did with log expressions in the previous task and soliciting a few exponential and log statements like this:

$$5^3 = 125 \quad so \quad \log_5 125 = 3$$

Encourage the use of different bases to remind students that the same definition works for all bases, b > 1. Tell students that in this task they will use what they know about inverses to help them create tables and graph log functions.

Explore (Small Group): Monitor students as they work to see that they are completing both tables and graphs for each function. Some students may choose to graph to exponential function and then reflect it over the y = x line to get the graph before completing the table. Watch for this strategy and be prepared to highlight it during the discussion. Finding points on the graph for $0 < x < 1$ may prove difficult for students since negative exponents are often difficult. Remind them that it may be easier to find points on the exponential function and then switch them for the log graphs if they are stuck.

Discuss (Whole Class): Begin the discussion with the graph of $f(x) = \log_2 x$. Ask a student that used the exponential function $y = 2^x$ and switched the x and y values to present their graph. Then have a student that started by creating a table describe how they obtained the values in the table. Ask the class to identify how the two strategies are connected. By now, students should be able to articulate the idea that powers of 2 are easy values to think about and that the value of the log expression will be the exponent in each case.

Move the discussion to question #4, the similarities between the graphs. Students will probably speak generally about the shapes being alike. In the discussions of similarities, be sure that the more technical features of the graphs emerge:

- The point (1,0) is included
- The domain is $(0, \infty)$
- The range is $(-\infty,\infty)$
- The function is increasing over the entire domain.

Ask students to connect each of these features will the definition of a logarithm and properties of inverse functions.

Ask what conclusions they could draw about the effect of changing the base on graph. How do these conclusions connect to the strategies they used to order log expressions with different bases in the previous task?

Ask students how the graphs were transformed when a number is added outside the log functions versus inside the argument of the log function. Students should notice that this is just like other functions that they are familiar with such as quadratic functions.

Conclude the discussion by having students present their work for 7a and 8b.

Aligned Ready, Set, Go: ***Logarithmic Functions 2.2***

Ready, Set, Go!

Ready

Topic: Solving simple logarithmic equations

Find the answer to each logarithmic equation. Then explain how each equation supports the statement, *"The answer to a logarithmic equation is always the exponent."*

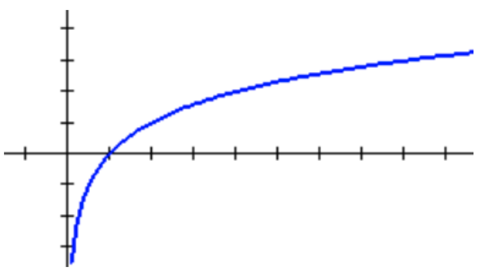
1. $log_5 625 =$
2. $log_3 243 =$
3. $log_5 0.2 =$
4. $log_9 81 =$
5. $log 1{,}000{,}000 =$
6. $log_x x^7 =$

Set

Topic: Transformations on logarithmic functions

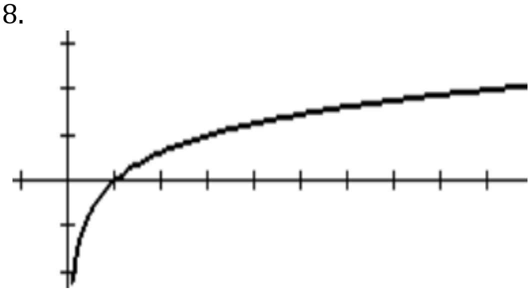
Answer the questions about each graph. (You may want to use a straightedge to find $f(x)$.

7.



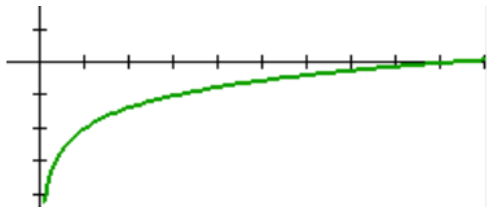
a. What is the value of x when $f(x) = 0$?
b. What is the value of x when $f(x) = 1$?
c. What is the value of $f(x)$ when $x = 2$?
d. What will be the value of x when $f(x) = 4$?
e. What is the equation of this graph?

8.



f. What is the value of x when $f(x) = 0$?
g. What is the value of x when $f(x) = 1$?
h. What is the value of $f(x)$ when $x = 9$?
i. What will be the value of x when $f(x) = 4$?
j. What is the equation of this graph?

9.

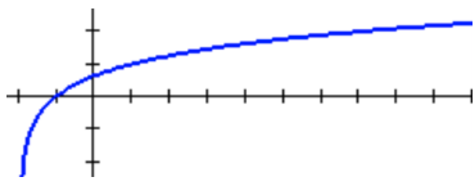


Use the graph and the table of values for the graph to write the equation of the graph.

Explain which numbers in the table helped you the most to write the equation.

X	Y1
1	-2
2	-1.369
3	-1
5	-.535
7	-.2288
9	0

10.



Use the graph and the table of values for the graph to write the equation of the graph.

Explain which numbers in the table helped you the most to write the equation.

X	Y1
-2	ERROR
-1	0
0	.63093
1	1
5	1.7712
6	1.8928
7	2

Go

Topic: Power to a power rule with exponents

Simplify each expression. Answers should have only positive exponents.

11. $(2^3)^4$

12. $(x^3)^2$

13. $(a^3)^{-2}$

14. $(2^3w)^4$

15. $(b^{-7})^3$

16. $(d^{-3})^{-2}$

17. $x^2 \cdot (x^5)^2$

18. $m^{-3} \cdot (m^2)^3$

19. $(x^5)^{-4} \cdot x^{25}$

20. $(5a^3)^2$

21. $(6^{-3})^2$

22. $(2a^3b^2)^2$

2.3 Chopping Logs
A Solidify Understanding Task

Abe and Mary are working on their math homework together when Abe has a brilliant idea!

Abe: I was just looking at this log function that we graphed in Falling Off A Log:

$$y = \log_2(x + b).$$

I started to think that maybe I could just "distribute" the log so that I get:

$$y = \log_2 x + \log_2 b.$$

I guess I'm saying that I think these are equivalent expressions, so I could write it this way:

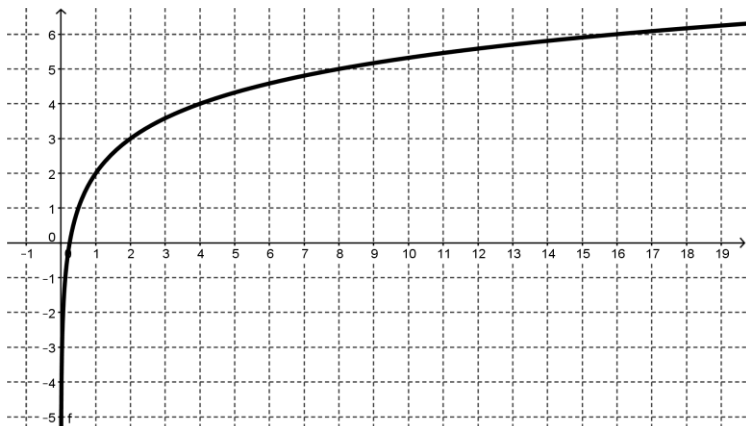
$$\log_2(x + b) = \log_2 x + \log_2 b$$

Mary: I don't know about that. Logs are tricky and I don't think that you're really doing the same thing here as when you distribute a number.

1. What do you think? How can you verify if Abe's idea works?

2. If Abe's idea works, give some examples that illustrate why it works. If Abe's idea doesn't work, give a counter-example.

Abe: I just know that there is something going on with these logs. I just graphed $f(x) = \log_2(4x)$. Here it is:



It's weird because I think that this graph is just a translation of $y = \log_2 x$. Is it possible that the equation of this graph could be written more than one way?

3. How would you answer Abe's question? Are there conditions that could allow the same graph to have different equations?

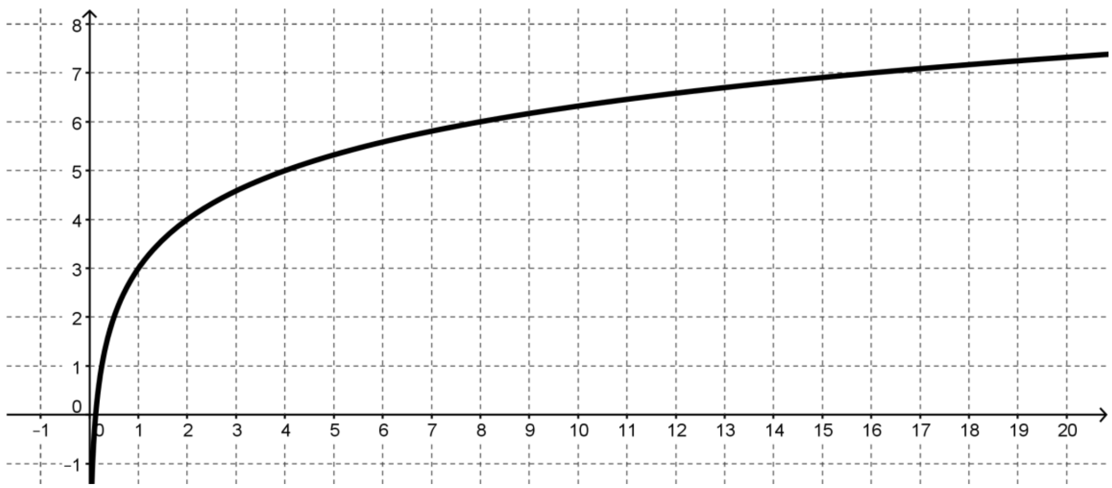
Mary: When you say, "a translation of $y = \log_2 x$" do you mean that it is just a vertical or horizontal shift? What could that equation be?

4. Find an equation for $f(x)$ that shows it to be a horizontal or vertical shift of $y = \log_2 x$.

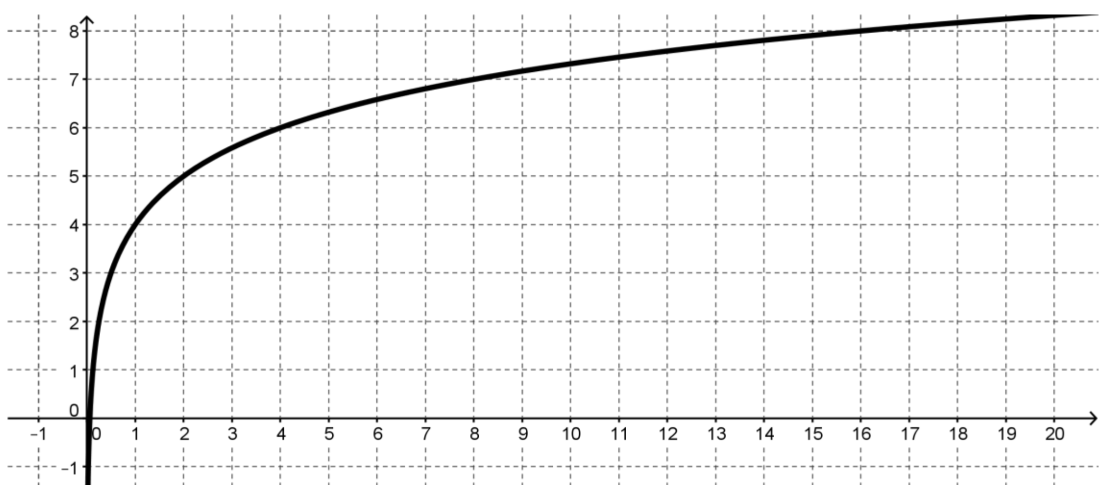
Mary: I wonder why the vertical shift turned out to be up 2 when the x was multiplied by 4. I wonder if it has something to do with the power that the base is raised to, since this is a log function. Let's try to see what happens with $y = \log_2(8x)$ and $y = \log_2(16x)$.

5. Try to write an equivalent equation for each of these graphs that is a vertical shift of $y = \log_2 x$.

a) $y = \log_2(8x)$ Equivalent equation: ____________________



b. $y = \log_2(16x)$ Equivalent equation: ____________________



Mary: Oh my gosh! I think I know what is happening here! Here's what we see from the graphs:

$$\log_2(4x) = 2 + \log_2 x$$

$$\log_2(8x) = 3 + \log_2 x$$

$$\log_2(16x) = 4 + \log_2 x$$

Here's the brilliant part: We know that $\log_2 4 = 2, \log_2 8 = 3,$ and $\log_2 16 = 4$. So:

$$\log_2(4x) = \log_2 4 + \log_2 x$$

$$\log_2(8x) = \log_2 8 + \log_2 x$$

$$\log_2(16x) = \log_2 16 + \log_2 x$$

I think it looks like the "distributive" thing that you were trying to do, but since you can't really distribute a function, it's really just a log multiplication rule. I guess my rule would be:

$$\log_2(ab) = \log_2 a + \log_2 b$$

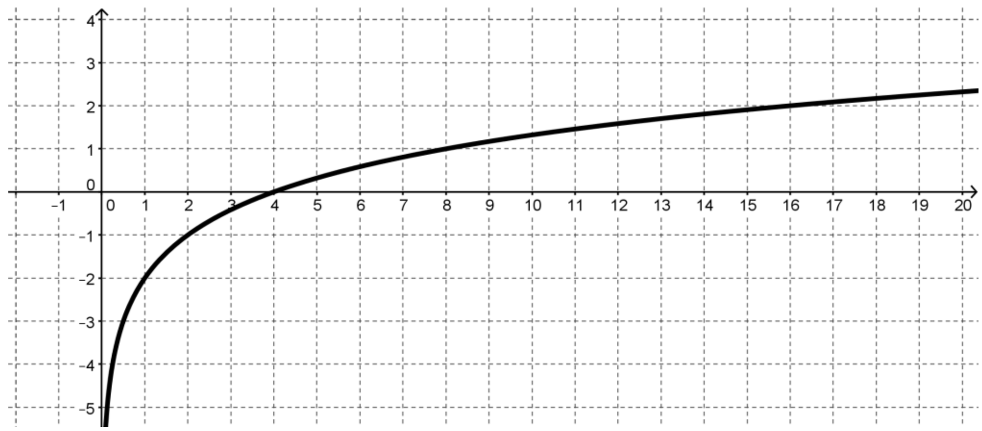
6. How can you express Mary's rule in words?

7. Is this statement true? If it is, give some examples that illustrate why it works. If it is not true provide a counter example.

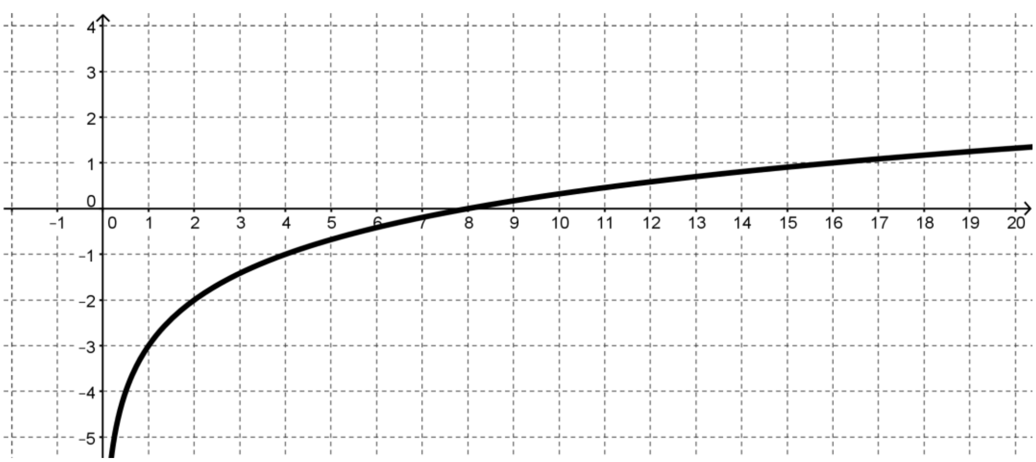
Mary: So, I wonder if a similar thing happens if you have division inside the argument of a log function. I'm going to try some examples. If my theory works, then all of these graphs will just be vertical shifts of $y = \log_2 x$.

8. Here are Abe's examples and their graphs. Test Abe's theory by trying to write an equivalent equation for each of these graphs that is a vertical shift of $y = \log_2 x$.

a) $y = \log_2\left(\frac{x}{4}\right)$ Equivalent equation: ____________________



b) $y = \log_2\left(\frac{x}{8}\right)$ Equivalent equation: ____________________



9. Use these examples to write a rule for division inside the argument of a log that is like the rule that Mary wrote for multiplication inside a log.

10. Is this statement true? If it is, give some examples that illustrate why it works. If it is not true provide a counter example.

Abe: You're definitely brilliant for thinking of that multiplication rule. But I'm a genius because I've used your multiplication rule to come up with a power rule. Let's say that you start with:

$$\log_2(x^3)$$

Really that's the same as having: $\log_2(x \cdot x \cdot x)$

So, I could use your multiplying rule and write: $\log_2 x + \log_2 x + \log_2 x$

I notice that there are 3 terms that are all the same. That makes it: $3\log_2 x$

So my rule is: $\log_2(x^3) = 3\log_2 x$

If your rule is true, then I have proven my power rule.

Mary: I don't think it's really a power rule unless it works for any power. You only showed how it might work for 3.

Abe: Oh good grief! Ok, I'm going to say that it can be any number x, raised to any power, k. My power rule is:

$$\log_2(x^k) = k\log_2 x$$

Are you satisfied?

11. Make an argument about Abe's power rule. Is it true or not?

Abe: Before we win the Nobel Prize for mathematics I suppose that we need to think about whether or not these rules work for any base.

12. The three rules, written for any base $b > 1$ are:

    Log of a Product Rule: $\log_b(xy) = \log_b x + \log_b y$

    Log of a Quotient Rule: $\log_b\left(\frac{x}{y}\right) = \log_b x - \log_b y$

    Log of a Power Rule: $\log_b(x^k) = k\log_b x$

    Make an argument for why these rules will work in any base $b > 1$ if they work for base 2.

13. How are these rules similar to the rules for exponents? Why might exponents and logs have similar rules?

Chopping Logs – Teacher Notes

A Solidify Understanding Task

Purpose: The purpose of this task is to use student understanding of log graphs and log expressions to derive properties of logarithms. In the task students are asked to find equivalent equations for graphs and then to generalize the patterns to establish the product, quotient, and power rules for logarithms.

Core Standards Focus:

F.IF.8. Write a function defined by an expression in different but equivalent forms to reveal and explain different properties of the function.

F.LE.4. For exponential models, express as a logarithm the solution to $ab^{ct} = d$ where a, c, and d are numbers and the base b is 2, 10, or e; evaluate the logarithm using technology.

Note to F.LE.4: *Consider extending this unit to include the relationship between properties of logarithms and properties of exponents, such as the connection between the properties of exponents and the basic logarithm property that log(xy) = log x + log y.*

Related Standards: F.BF.5

Note: The nature of this task suggests a more guided approach from the teacher than many tasks. The most productive classroom configuration might be pairs, so that students can easily shift their attention back and forth from whole group discussion to their own work.

Launch (Whole Class): Begin the task by introducing the equation: $\log_2(x+b) = \log_2 x + \log_2 b$. Ask students why this might make sense. Expect to hear that they have "distributed" the log. Without judging the merits of this idea, ask students how they could test the claim. When the idea to test particular numbers comes up, set students to work on questions #1 and #2. After students have had a chance to work on #2, ask a student that has an example that shows the statement to be untrue to share his/her work. You may need to help rewrite the student's work so that the statements are clear because this is a strategy that students will want to use throughout the task.

Explore (Small Group): Ask students to turn their attention to question #3. Based on their work with log graphs in previous tasks, they have not seen the graph of a log function with a multiplier in the argument, like $f(x) = \log_2(4x)$. Ask students how the 4 might affect the graph. Because Abe thinks that the function is a vertical shift of $y = \log_2 x$, ask student what the equation would look like with a vertical shift so that they generate the idea that the vertical shift typically looks like $y = a + \log_2 x$. Ask students to investigate questions #3 and #4. As students are working, look for a student that has redrawn the x-axis or use a straight edge to show the translation of the graph.

Discuss (Whole Class): When students have had enough to time to find the vertical shift, have a student demonstrate how they were able to tell that the function is a vertical shift up 2. It will help move the class forward in the next part of the task if a student demonstrates redrawing the x-axis so it is easy to see that the graph is a translation, but not a dilation of $y = \log_2 x$. After this discussion, have students work on questions 5, 6, and 7.

Explore (Small Group): While students are working, listen for students that are able to describe the pattern that Mary has noticed in the task. Encourage students to test Mary's conjecture will some numbers, just as they did in the beginning of the task.

Discuss (Whole Class): Ask several students to state Mary's rule in their own words. Try to combine the student statements into something like: "The log of a product is the sum of the logs" or give them this statement and ask them to discuss how it describes the pattern that they have noticed. Have some students show some examples that provide evidence that the statement is true, but remind students that a few examples don't count as a proof. After this discussion, ask students to complete the rest of the task.

Explore (Small Group): Support students as they work to recognize the patterns and express a rule in #9 as both an equation and in words. Students may have difficulty with the notation, so ask them to state the rule in words first, and then help them to write it symbolically.

Discuss (Whole Class): The remaining discussion should follow each of the questions in the task from #9 onward. As the discussion progresses, show student examples of each rule, both to provide evidence that the rule is true and also to practice using the rule. Emphasize reasoning that helps students to see that the log rules are like the exponent rules because of the relationship between logs and exponents.

Aligned Ready, Set, Go: ***Logarithmic Functions 2.3***

Name Logarithmic Functions | 2.3

Ready, Set, Go!

Ready

Topic: Fractional exponents

Write the following with an exponent. Simplify when possible.

1. $\sqrt[5]{x}$ 2. $\sqrt[7]{s^2}$ 3. $\sqrt[3]{w^8}$

4. $\sqrt[3]{8r^6}$ 5. $\sqrt[5]{125m^5}$ 6. $\sqrt[3]{(8x)^2}$ 7. $\sqrt[3]{9b^8}$ 8. $\sqrt{75x^6}$

Rewrite with a fractional exponent. Then find the answer.

9. $log_3\sqrt[5]{3} =$ 10. $log_2\sqrt[3]{4} =$ 11. $log_7\sqrt[5]{343} =$ 12. $log_5\sqrt[5]{3125} =$

Set

Topic: Expanding logarithmic expressions

Use the properties of logarithms to expand the expression as a sum or difference, and/or constant multiple of logarithms. (Assume all variables are positive.)

13. $log_5 7x$ 14. $log_5 10a$ 15. $log_5 \frac{5}{b}$ 16. $log_5 \frac{d}{4}$

17. $log_6 x^3$

18. $log_5 9x^2$

19. $log_2 (7x)^4$

20. $log_3 \sqrt{w}$

21. $log_5 \frac{xyz}{w}$

22. $log_5 \frac{9\sqrt{x}}{y^3}$

23. $log_2 \left(\frac{x^2-4}{x^3}\right)$

24. $log_2 \left(\frac{x^2}{y^5 w^3}\right)$

Go

Topic: Writing expressions in exponential form and logarithmic form

Convert to logarithmic form.

25. $2^9 = 512$

26. $10^{-2} = 0.01$

27. $\left(\frac{2}{3}\right)^{-1} = \frac{3}{2}$

Write in exponential form.

28. $log_4 2 = \frac{1}{2}$

29. $log_{\frac{1}{3}} 3 = -1$

30. $log_{\frac{2}{5}} \frac{8}{125} = 3$

2.4 Log-Arithm-etic
A Practice Understanding Task

Abe and Mary are feeling good about their log rules and bragging about mathematical prowess to all their friends when this exchange occurs:

Stephen: I guess you think you're pretty smart because you figured out some log rules, but I want to know what they're good for.

Abe: Well, we've seen a lot of times when equivalent expressions are handy. Sometimes when you write an expression with a variable in it in a different way it means something different.

1. What are some examples from your previous experience where equivalent expressions were useful?

Mary: I was thinking about the Log Logic task where we were trying to estimate and order certain log values. I was wondering if we could use our log rules to take values we know and use them to find values that we don't know.

For instance: Let's say you want to calculate $\log_2 6$. If you know what $\log_2 2$ and $\log_2 3$ are then you can use the product rule and say:

$$\log_2(2 \cdot 3) = \log_2 2 + \log_2 3$$

Stephen: That's great. Everyone knows that $\log_2 2 = 1$, but what is $\log_2 3$?

Abe: Oh, I saw this somewhere. Uh, $\log_2 3 = 1.585$. So Mary's idea really works. You say:

$$\log_2(2 \cdot 3) = \log_2 2 + \log_2 3$$

$$= 1 + 1.585$$

$$= 2.585$$

$$\log_2 6 = 2.585$$

2. Based on what you know about logarithms, explain why 2.585 is a reasonable value for $\log_2 6$.

Stephen: Oh, oh! I've got one. I can figure out $\log_2 5$ like this:

$$\log_2(2+3) = \log_2 2 + \log_2 3$$

$$= 1 + 1.585$$

$$= 2.585$$

$$\log_2 5 = 2.585$$

3. Can Stephen and Mary both be correct? Explain who's right, who's wrong (if anyone) and why.

Now you can try applying the log rules yourself. Use the values that are given and the ones that you know by definition like $\log_2 2 = 1$ to figure out each of the following values. Explain what you did in the space below each question.

$\log_2 3 = 1.585$ $\quad\quad$ $\log_2 5 = 2.322$ $\quad\quad$ $\log_2 7 = 2.807$

The three rules, written for any base $b > 1$ are:

Log of a Product Rule: $\quad \boldsymbol{\log_b(xy) = \log_b x + \log_b y}$

Log of a Quotient Rule: $\quad \boldsymbol{\log_b\left(\frac{x}{y}\right) = \log_b x - \log_b y}$

Log of a Power Rule: $\quad \boldsymbol{\log_b(x^k) = k\log_b x}$

4. $\log_2 9 =$ ____

5. $\log_2 10 =$ ____

6. $\log_2 12 =$ ______________________________

7. $\log_2 \left(\frac{7}{3}\right) =$ ______________________________

8. $\log_2 \left(\frac{30}{7}\right) =$ ______________________________

9. $\log_2 486 =$ ______________________________

10. Given the work that you have just done, what other values would you need to figure out the value of the base 2 log for any number?

Sometimes thinking about equivalent expressions with logarithms can get tricky. Consider each of the following expressions and decide if they are always true for the numbers in the domain of the logarithmic function, sometimes true, or never true. Explain your answers. If you answer "sometimes true" then describe the conditions that must be in place to make the statement true.

11. $\log_4 5 - \log_4 x = \log_4 \left(\frac{5}{x}\right)$ ______________________________

12. $\log 3 - \log x - \log x = \log \left(\frac{3}{x^2}\right)$ ______________________________

13. $\log x - \log 5 = \frac{\log x}{\log 5}$ ______________________________

14. $5 \log x = \log x^5$ ______________________________

15. $2 \log x + \log 5 = \log(x^2 + 5)$ ______________________________

16. $\frac{1}{2} \log x = \log \sqrt{x}$ ______________________________

17. $\log(x - 5) = \frac{\log x}{\log 5}$ ______________________________

Log-Arithm-etic – Teacher Notes

A Practice Understanding Task

Purpose: The purpose of this task is to extend student understanding of log properties and using the properties to write equivalent expressions. In the beginning of the task, students are asked to use log properties, given values of a few log expressions, and known values of log expressions to find unknown values. This is an opportunity to see how the known log values can be used and to practice using logarithms and substitution. In the second part of the task, students are asked to determine if the given equations are always true (in the domain of the expression), sometimes true, or never true. This gives students an opportunity to work through some common misconceptions about log properties and to write equivalent expressions using logs.

Core Standards Focus:

F.IF.8. Write a function defined by an expression in different but equivalent forms to reveal and explain different properties of the function.

F.LE.4. For exponential models, express as a logarithm the solution to $ab^{ct} = d$ where a, c, and d are numbers and the base b is 2, 10, or e; evaluate the logarithm using technology.

Note to F.LE.4: *Consider extending this unit to include the relationship between properties of logarithms and properties of exponents, such as the connection between the properties of exponents and the basic logarithm property that log(xy) = log x + log y.*

Launch (Whole Class): Launch the task by reading through the scenario and asking students to work problem 1. Follow it by a discussion of their answers, pointing out that equivalent forms often have different meanings in a story context and that they can be helpful in solving equations and graphing. Follow this short discussion by having students work problems 2 and 3, then discussing them as a class. The purpose of question #2 is to demonstrate how to use the log rules to find values, and emphasize how they can use the definition of a logarithm to determine if the value they find is reasonable. After discussing these two problems, students should be ready to use the properties to find values of logs. At this point, have students work questions 4-10 before coming back for a discussion.

Explore (Small Group): As students are working, they may need support in finding combinations of factors to use so that they can apply the log properties. You may want to remind them of using factor trees or a similar strategy for breaking down a number into its factors. Watch for two students that use different combination of factors to find the value they are looking for. As you are monitoring student work, be sure that they are using good notation to communicate how they are finding the values.

Discuss (Whole Class): Discuss a few of the problems, selecting those that caused controversy among students as they worked. For each problem, be sure to demonstrate the way to use notation and the log properties to find the values. An example might be:

$$\log_2\left(\frac{30}{7}\right) = \log_2 30 - \log_2 7$$

$$= \log_2(5 \cdot 2 \cdot 3) - \log_2 7$$

$$= \log_2 5 + \log_2 2 + \log_2 3 - \log_2 7$$

$$= 2.322 + 1 + 1.585 - 2.807$$

$$=2.1$$

After finding each value, discuss whether or not the answer is reasonable. After a few of these problems, turn students' attention to the remainder of the task.

Explore (Small Group): Support students as they work in making sense of the statements and verifying them. The statements are designed to bring out misconceptions, so discussion among students should be encouraged. There are several possible strategies for verifying these equations, including using the log properties to manipulate one side of the equation to match the other or trying to put in numbers to the statement. Look for both types of strategies so that the numerical approach can provide evidence, but the algebraic approach can prove (or disprove) the statement.

Discuss (Whole Group): Again, select problems for discussion that have generated controversy or exposed misconceptions. It will often be useful to test the statement with numbers, although that may be difficult for students in some cases. Encourage students to cite the log property that they are using as they manipulate the statements to show equivalence.

11. Always true 12. Always true 13. Never true

14. Always true 15. Never true 16. Always true

17. Never true

Aligned Ready, Set, Go: ***Logarithmic Functions 2.4***

Name Logarithmic Functions | 2.4

Ready, Set, Go!

Ready

Topic: Solving simple exponential and logarithmic equations

You have solved exponential equations before based on the idea that
$\boldsymbol{a^x = a^y, if\ and\ only\ if\ x = y}$.

You can use the same logic on logarithmic equations. $\boldsymbol{log_a x = log_b y, if\ and\ only\ if\ x = y}$
Rewrite each equation so that you set up a one-to-one correspondence between all of the parts. Then solve for x.

Example: ***Original equation:***	***Rewritten equation:***	***Solution:***
a.) $3^x = 81$	$3^x = 3^4$	$x = 4$
b.) $log_2 x - log_2 5 = 0$	$log_2 x = log_2 5$	$x = 5$

1. $3^{x+4} = 243$

2. $\left(\frac{1}{2}\right)^x = 8$

3. $\left(\frac{3}{4}\right)^x = \frac{27}{64}$

4. $log_2 x - log_2 13 = 0$

5. $log_2(2x - 4) - log_2 8 = 0$

6. $log_2(x + 2) - log_2 9x = 0$

7. $\frac{\log 2x}{\log 14} = 1$

8. $\frac{\log(5x-1)}{\log 29} = 1$

9. $\frac{\log 5^{(x-2)}}{\log 625} = 1$

Set

Topic: Rewriting logs in terms of known logs

Use the given values and the properties of logarithms to find the indicated logarithm. Do not use a calculator to evaluate the logarithms.

Given: $\mathbf{\log 16 \approx 1.2}$

$\mathbf{\log 5 \approx 0.7}$

$\mathbf{\log 8 \approx 0.9}$

10. Find $\log \frac{5}{8}$

11. Find $\log 25$

12. Find $\log \frac{1}{2}$

13. Find $\log 80$

14. Find $\log \frac{1}{49}$

Given $\boldsymbol{log_3 2 \approx 0.6}$

$\boldsymbol{log_3 5 \approx 1.5}$

15. Find $log_3 16$

16. Find $log_3 108$

17. Find $log_3 \frac{3}{50}$

18. Find $log_3 \frac{8}{15}$

19. Find $log_3 486$

20. Find $log_3 18$

21. Find $log_3 120$

22. Find $log_3 \frac{32}{45}$

Go

Topic: Using the definition of logarithm to solve for x.

Use your calculator and the definition of $log x$ (recall: the base is 10) to find the value of x.
(Round your answers to 4 decimals.)

23. $\log x = -3$

24. $\log x = 1$

25. $\log x = 0$

26. $\log x = \frac{1}{2}$

27. $\log x = 1.75$

28. $\log x = -2.2$

29. $\log x = 3.67$

30. $\log x = \frac{3}{4}$

31. $\log x = 6$

2.5 Powerful Tens

A Practice Understanding Task

Table Puzzles

1. Use the tables to find the missing values of x:

a.

x	$y = 10^x$
-2	$\frac{1}{100}$
1	10
	50
	100
3	1000

b.

x	$y = 3(10^x)$
	0.3
0	3
	94.87
2	300
	1503.56

c. What equations could be written, in terms of x only, for each of the rows that are missing the x in the two tables above?

d.

x	$y = \log x$
0.01	-2
	-1
10	1
	1.6
100	2

e.

x	$y = \log(x+3)$
	-2
-2.9	-1
	0.3
7	1
	3

f. What equations could be written, in terms of *x* only, for each of the rows that are missing the *x* in the two tables above?

2. What strategy did you use to find the solutions to equations generated by the tables that contained exponential functions?

3. What strategy did you use to find the solutions to equations generated by the tables that contained logarithmic functions?

Graph Puzzles

4. The graph of y= 10^{-x} is given below. Use the graph to solve the equations for *x* and label the solutions.

a. $40 = 10^{-x}$

$x =$ _____

Label the solution with an A on the graph.

b. $10^{-x} = 10$

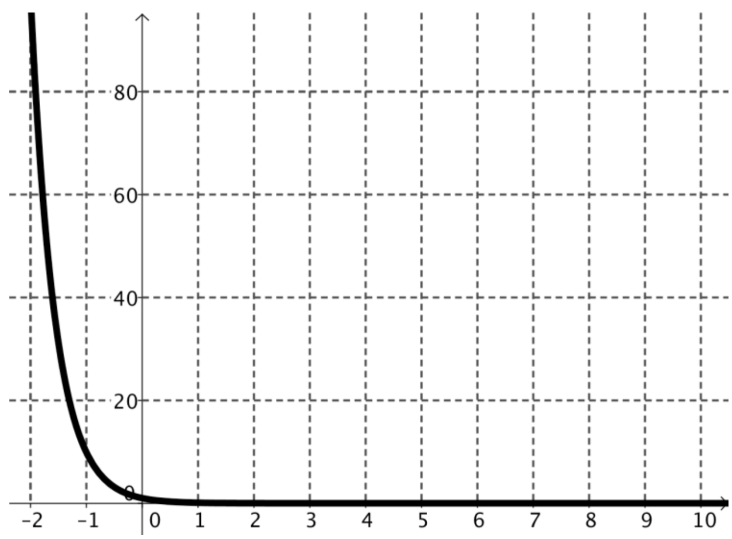
$x =$ _____

Label the solution with a B on the graph.

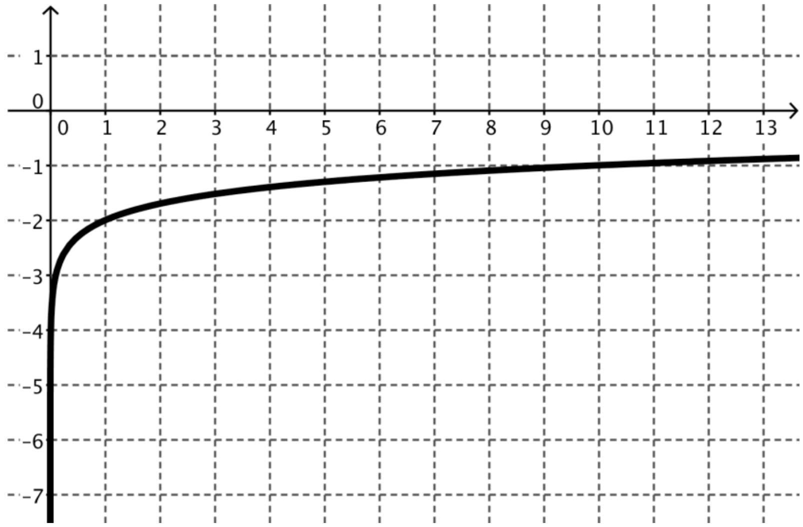
c. $10^{-x} = 0.1$

$x =$ _____

Label the solution with a C on the graph.



5. The graph of y= $-2 + \log x$ is given below. Use the graph to solve the equations for x and label the solutions.



a. $-2 + \log x = -2$

$x =$ _____

Label the solution with an A on the graph.

b. $-2 + \log x = 0$

$x =$ _____

Label the solution with a B on the graph.

c. $-4 = -2 + \log x$

$x =$ _____

Label the solution with a C on the graph.

d. $-1.3 = -2 + \log x$

$x =$ _____

Label the solution with a D on the graph.

e. $1 = -2 + \log x$

$x =$ _____

6. Are the solutions that you found in #5 exact or approximate? Why?

Equation Puzzles:

Solve each equation for *x:*

7. 10^x=10,000

8. $125 = 10^x$

9. $10^{x+2} = 347$

10. $5(10^{x+2}) = 0.25$

11. $10^{-x-1} = \frac{1}{36}$

12. $-(10^{x+2}) = 16$

2.5 Powerful Tens – Teacher Notes

A Practice Understanding Task

Note: Calculators or other technology with base 10 logarithmic and exponential functions are required for this task.

Purpose:
The purpose of this task is to develop student ideas about solving exponential equations that require the use of logarithms and solving logarithmic equations. The task begins with students finding unknown values in tables and writing the corresponding equation for equations. In the second part of the task, students use graphs to find equation solutions. Finally, students build on their thinking with tables and graphs to solve equations algebraically. All of the logarithmic and exponential equations are in base 10 so that students can use technology to find values.

Core Standards Focus:

F.LE.4. For exponential models, express as a logarithm the solution to $ab^{ct} = d$ where a, c, and d are numbers and the base b is 2, 10, or e; evaluate the logarithm using technology.

Launch (Whole Class):

Remind students that they are very familiar with constructing tables for various functions. In previous tasks, they have selected values for *x* and calculated the value of *y* based upon an equation or other representation. They have also constructed graphs based upon having an equation or a set of *x* and *y* values. In this task they will be using tables and graphs to work in reverse, finding the *x* value for a given *y*.

Explore (Small Group):

Monitor students as they work and listen to their strategies for finding the missing values of *x*. As they are working on the table puzzles, encourage them to consider writing equations as a way to track their strategies. In the graph puzzles, they will find that they can only get approximate answers on a few equations. Encourage them to use the graph to estimate a value and to interpret the solution in the equation. The purpose of the tables and graphs is to help students draw upon their thinking from previous tasks to solve the equations. Remind students to connect the ideas as they work on the equation puzzles.

Discuss (Whole Class):

Start the discussion with a student that has written and solved an equation for the third row in table b. The equation written should be:

$$\mathbf{94.87 = 3(10^{x})}$$

Ask the student to describe how they wrote the equation and then their strategies for solving it. Be sure to have students describe their thinking about how to unwind the function as the steps are tracked on the equation. Ask the class where this point would be on a graph of the function. Ask students what the graph of the function would look like, and they should be able to describe a base 10 exponential function with a dilation or vertical stretch of 3.

Move the discussion to table e, focusing on the last row of the table. Again, have students write the equation:

$$\mathbf{3 = \log(x + 3)}$$

Ask the presenting student to describe his/her thinking in how to find the value of x in the table and once again, track the steps algebraically. There are a couple of likely mistakes made by students that have tried to solve this equation algebraically. If they arise during your observation of students, discuss them here. Again, connect the solution they found to the graph of the function. Students should be noticing that since logs and exponentials are inverse functions, exponential equations can be solved with logs and log equations are solved with exponentials.

Move the discussion to the graph of $y = 10^{-x}$. Ask students to describe how they used the graph to find the solution to "a". Ask students how they could check the solution in the equation. Does the solution they found with the graph make sense? How would they solve this equation without a graph? Track the steps algebraically, showing something like the following:

$$40 = 10^{-x}$$

$$\log 40 = \log(10^{-x})$$

$$1.602 = -x$$

(Make sure students can explain this step, both using the calculator and simplifying the right side of the equation. It would be useful if students noticed that they could use the log properties to rewrite the right side of the equation as $-x(\log 10)$ in addition to using the definition of the logarithm.)

$$x = -1.602$$

Finally, ask students to show solutions to as many of the equation puzzles that time will allow. In every case, be sure that students can describe how they use logs to undo the exponential and that their notation matches their thinking.

Aligned Ready, Set, Go: *Logarithmic Functions 2.5*

Name Logarithmic Functions 2.5

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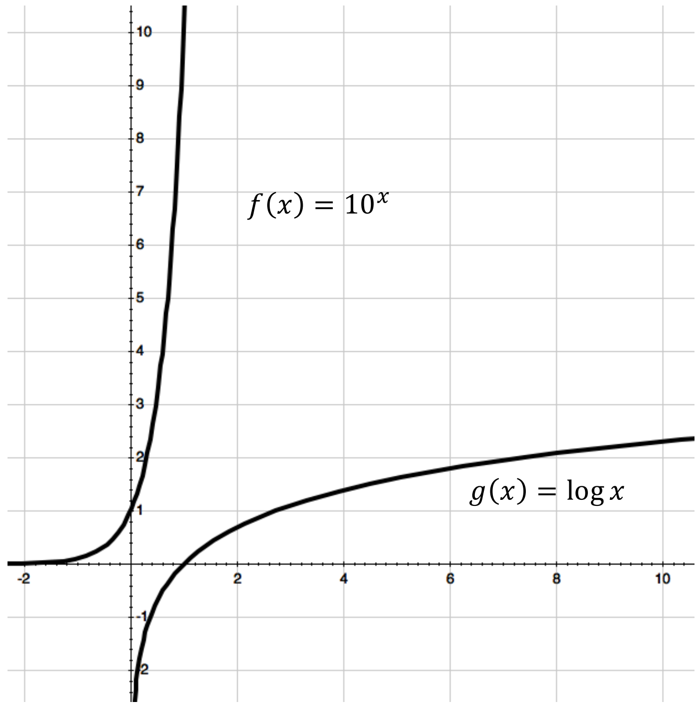
Ready, Set, Go!

Ready

Topic: Comparing the exponential and logarithmic graphs

The graphs of $f(x) = 10^x$ *and* $g(x) = \log x$ are shown in the same coordinate plane. Make a list of the characteristics of each function.



1. $f(x) = 10^x$

2. $g(x) = \log x$

Each question below refers to the graphs of the functions $f(x) = 10^x$ *and* $g(x) = \log x$. State whether they are true or false. If they are false, correct the statement so that it is true.

_________ 3. Every graph of the form $g(x) = \log x$ will contain the point (1, 0).

_________ 4. Both graphs have vertical asymptotes.

_________ 5. The graphs of $f(x)$ *and* $g(x)$ have the same rate of change.

_________ 6. The functions are inverses of each other.

_________ 7. The range of $f(x)$ is the domain of $g(x)$.

_________ 8. The graph of $g(x)$will never reach 3.

Name Logarithmic Functions 2.5

Set

Topic: Solving logarithmic equations base 10 by taking the log of each side.

Evaluate the following logarithms.

9. $\log 10$
10. $\log 10^{-7}$
11. $\log 10^{75}$
12. $\log 10^{x}$

13. $log_3 3^5$
14. $log_8 8^{-3}$
15. $log_{11} 11^{37}$
16. $log_m m^n$

You can use this property of logarithms to help you solve logarithmic equations. *Note that this property only works when the base of the logarithm matches the base of the exponent.*

Solve the equations by inserting $\boldsymbol{log_m}$ on both sides of the equation. (You will need a calculator.)

17. $10^n = 4.305$
18. $10^n = 0.316$
19. $10^n = 14{,}521$
20. $10^n = 483.059$

Go

Topic: Solving equations involving compound interest

Formula for compound interest: If P dollars is deposited in an account paying an annual rate of interest r compounded (paid) n times per year, the account will contain $\boldsymbol{A = P\left(1 + \frac{r}{n}\right)^{nt}}$ dollars after t years.

21. How much money will there be in an account at the end of 10 years if \$3000 is deposited at 6% annual interest compounded as follows: (Assume no withdrawals are made.)
    a.) annually
    b.) semiannually
    c.) quarterly
    d.) daily (Use n = 365.)

22. Find the amount of money in an account after 12 years if \$5,000 is deposited at 7.5% annual interest compounded as follows: (Assume no withdrawals are made.)
    a.) annually
    b.) semiannually
    c.) quarterly
    d.) daily (Use n = 365.)