mathematics vision project

Transforming Mathematics Education

SECONDARY MATH THREE

An Integrated Approach

Standard Teacher Notes

MODULE 6

Modeling Periodic Behavior

MATHEMATICSVISIONPROJECT.ORG

The Mathematics Vision Project

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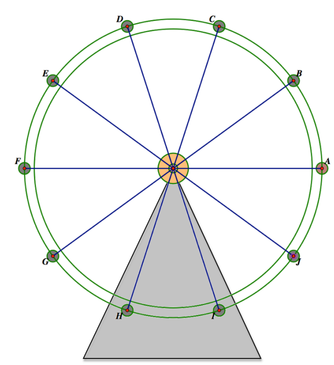
6.1 George W. Ferris' Day Off

A Develop Understanding Task

Perhaps you have enjoyed riding on a Ferris wheel at an amusement park. The Ferris wheel was invented by George Washington Ferris for the 1893 Chicago World's Fair.

Carlos, Clarita and their friends are celebrating the end of the school year at a local amusement park. Carlos has always been afraid of heights, and now his friends have talked him into taking a ride on the amusement park Ferris wheel. As Carlos waits nervously in line he has been able to gather some information about the wheel. By asking the ride operator, he found out that this wheel has a radius of 25 feet, and its center is 30 feet above the ground. With this information, Carlos is trying to figure out how high he will be at different positions on the wheel.



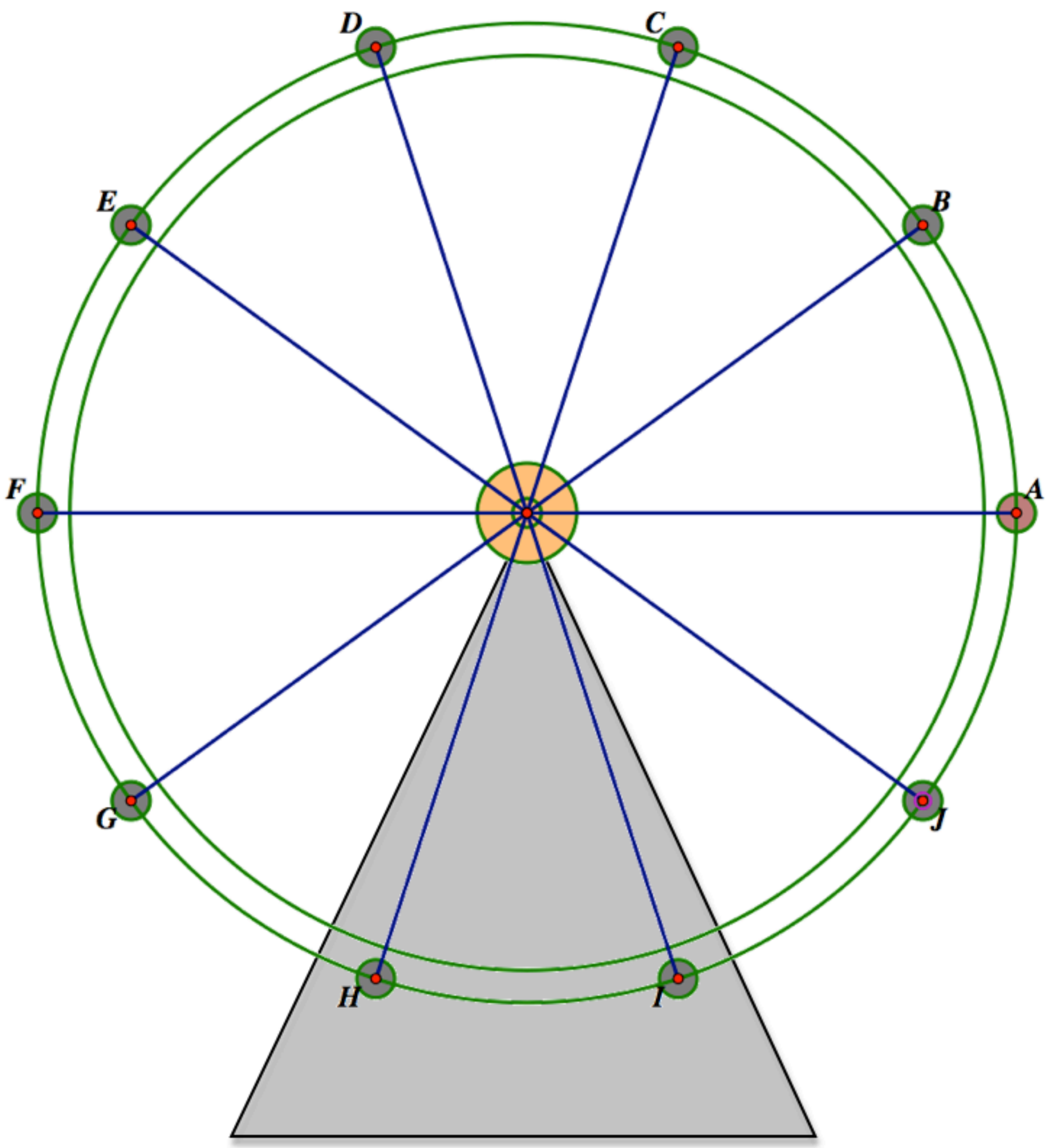
1. How high above the ground will Carlos be when he is at the top of the wheel? (To make things easier, think of his location as simply a point on the circumference of the wheel's circular path.)

2. How high will he be when he is at the bottom of the wheel?

3. How high will he be when he is at the positions farthest to the left or the right on the wheel?

Because the wheel has ten spokes, Carlos wonders if he can determine the height of the positions at the ends of each of the spokes as shown in the diagram on the following page. Carlos has just finished studying right triangle trigonometry, and wonders if that knowledge can help him.

4. Find the height of each of the points labeled A-J on the Ferris wheel diagram below. Represent your work on the diagram so it is apparent to others how you have calculated the height at each point.



6.1 George W. Ferris' Day Off – Teacher Notes

A Develop Understanding Task

Purpose: The purpose of this task is to help students visualize how right triangle trigonometric ratios can be used as a tool to describe periodic behavior—in this case, rotation around a Ferris wheel. In the next task, students will consider the motion of a point around a moving Ferris wheel. In this task, students consider points on a stationary Ferris wheel and determine how they can find the height of those points above the ground.

Imagining the spokes of the Ferris wheel as the hypotenuse of various right triangles—triangles drawn by dropping a vertical line from the endpoints of the spokes to the horizontal line through the center of the Ferris wheel—will allow students to use the sine ratio to find the distance of a point on the wheel above or below the center of the wheel. Students will develop a strategy for finding the height of any point: $height = 30 + 25\sin(\theta)$, where θ is the angle formed by the horizontal line through the center of the wheel to a particular spoke of the wheel. Since students are only familiar with right triangle trigonometry, θ is between 0° and 90°. Consequently, students will need to consider how to find θ when the endpoint of the spoke of interest lies in quadrants II, III or IV.

Students will also observe the symmetry of points around the wheel as a way to reduce the number of computations needed to find the heights of all ten endpoints. This task develops essential ways of thinking about the location of points around a circle, which will become fundamental in students' understanding of trigonometric functions, radian measure and the unit circle.

Core Standards Focus:

F.TF.5 Choose trigonometric functions to model periodic phenomena with specified amplitude, frequency, and midline.★

Related Standards: G.SRT.8

Standards for Mathematical Practice:
SMP 4 – Model with mathematics
SMP 7 – Look for and make use of structure

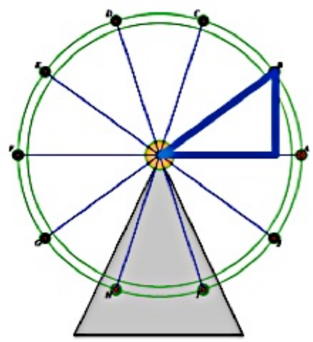
Vocabulary: Students will need to understand that the term *vertical height* refers to the distance a person is above the ground, and consists of two components—the distance above or below the center of the Ferris wheel, plus the distance from the ground to the center of the Ferris wheel.

The Teaching Cycle:
Launch (Whole Class):
After introducing the scenario from the introductory paragraphs of this task, ask students to individually answer questions 1, 2 and 3. After students have had a few minutes to think about the structure of the Ferris wheel, have a couple of students discuss how they could answer these questions. It is important to surface the idea that distances from the ground to any point on the Ferris wheel can be found by adding to or subtracting from the height of the center of the wheel above the ground.

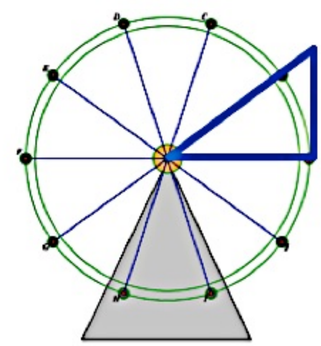
Remind students of what they know about right triangle trigonometry, particularly the definitions of the trigonometric ratios: for an acute angle of a right triangle, the *sine* of that angle is the ratio of the length of the side opposite the angle to the length of the hypotenuse; the *cosine* of that angle is the ratio of the length of the side adjacent to the length of the hypotenuse; and, the *tangent* of the angle is the ratio of the length of the side opposite to the length of the side adjacent. Once you have reviewed these definitions, set students to work on finding the distance from the ground to each of the endpoints of the ten spokes of the wheel.

Explore (Small Group):
Students may struggle with how to get started, since there are no right triangles drawn in the diagram. Let students struggle with this for a while, perhaps asking such questions as, “Where might you draw a right triangle on this diagram so you would know something about its sides or angles?” or “What do you know for sure about the triangle you just drew?”

Students may make several attempts at drawing right triangles until they find one for which they do know a length and an angle. Since it can be determined that the angle between two spokes measures 36° (one-tenth of the circle), and the length of a spoke is 25 feet (the radius of the wheel), there are two potential triangles students might draw for which they do know enough information to apply right triangle trigonometry, as shown in this diagram.

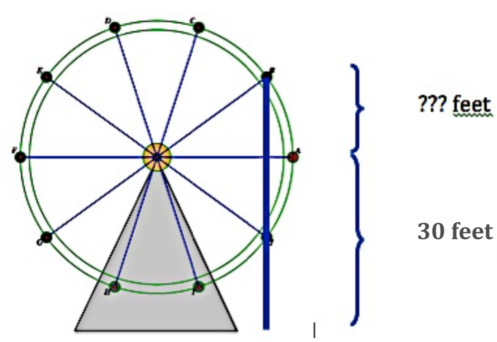


hypotenuse of known length

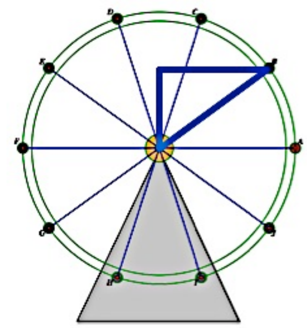


leg of known length

Students are often so pleased to find a triangle they can apply trigonometry to, that they do not recognize that the triangle that uses a spoke as a leg of the right triangle does not give them information about the height of an endpoint above or below the center of the wheel. Allow them to struggle with this idea, perhaps by asking the question, “And what would this triangle look like if you were to draw it for another endpoint?”



Since students know they want to find the distance from the ground to a point, watch for students that start by drawing a line segment from an endpoint of a spoke perpendicular to the ground. Listen for students who suggest they might decompose this line segment into two parts: the part which measures 30 feet from the ground to the line that passes horizontally through the center, and the part that lies above the horizontal line whose length will need to be calculated. Once students are focused on this segment that lies above the horizontal line through the center, suggest they try to draw a right triangle that includes this line segment. Watch to see if they draw the right triangle to include this line segment and the center of the wheel as one of its vertices. If so, this should lead them to a strategy that involves using the sine ratio since they will be calculating the length of a side opposite of the angle whose vertex is at the center of the Ferris wheel.



Some students may draw a triangle, such as shown in the diagram at the right, for which they can use the cosine ratio to find the distance above or below the center of the wheel. While this strategy works, it seems less straight forward than the strategy described in the previous paragraph, since we are determining the height of a point above the ground by comparing it to another point with the same height, and finding the measure of an acute angle in this triangle is not as straight forward as just noting the angle between the spokes.

Some students will change strategies from quadrant to quadrant, sometimes using a sine ratio in one quadrant and a cosine ratio in another. During the whole class discussion you will want to discuss the value of finding a single strategy that works in all cases, and help students recognize that using the sine ratio seems more straightforward in terms of the location of the point relative to the ground.

Discuss (Whole Class):

Have students share strategies for finding the height of each endpoint of the spokes on the Ferris wheel. Start by labeling the height of the points farthest to the right and left of the wheel—points A and F—since they lie along a horizontal line through the center of the wheel. Next, have someone present how they found the height at point B, and then have someone present their work for point C. Select students to present that used a sine ratio to calculate the height of each of these points.

As you move around the circle, select students to present who calculated the heights of points D, E, G, H, I and J, as well as students who used the symmetry of the circle to determine the heights. Lead students to generalize that the height of any endpoint of a spoke can be found using the formula *height* $= 30 + 25\sin(\theta)$, where θ is the angle formed by the horizontal line through the center of the wheel and the line segment that represent a particular spoke of the wheel.

If some students changed strategies from point to point or quadrant to quadrant, discuss the value of finding a single strategy that works in all cases. If some students used a cosine ratio, point out that it is a correct strategy, but suggest that using the sine ratio seems more straightforward in terms of the location of the point relative to the ground, since the endpoints of the spokes do not lie on the vertical line through the center of the wheel.

Aligned Ready, Set, Go: Modeling Periodic Behavior 6.1

READY, SET, GO! Name Period Date

READY

Topic: Making sense of speed when distance is circular

The number of degrees an object passes through during a given amount of time is called ***angular speed.*** For instance, the second hand on a clock has an angular speed of $\frac{360°}{min}$ while the minute hand on a clock has an angular speed of $\frac{360°}{hr}$. (Remember that a revolution is a full circle or 360º.)

1. What is the angular speed of the second hand on a clock in degrees per second?
2. What is the angular speed of the minute hand on a clock in degrees per second?
3. What is the angular speed of the hour hand in degrees per hour?

Your grandparents probably enjoyed music just as much as you do, but they didn't have IPods or MP3 players. They had vinyl records and phonographs. Vinyl records came in 3 speeds. A record could be a 45, $33\frac{1}{3}$, or 78. These numbers referred to the rpms or ***revolutions per minute***.

4. Calculate the ***angular speed*** of a $45\ rpm, 33\frac{1}{3}\ rpm, and\ 78\ rpm$ record in degrees per minute.
   a) $45\ rpm$
   b) $33\frac{1}{3}\ rpm$
   c) $78\ rpm$

Angular speed describes how fast something is turning. ***Linear speed*** describes how far it travels while it is turning. ***Linear speed*** depends on the circumference of a circle ($C = 2\pi r$) and the number of revolutions per minute.

Vinyl records were not the same size. A $45\ rpm$ record had a diameter of 7 inches, a $33\frac{1}{3}$ a diameter of 12 inches and a 78 had a diameter of 10 inches.

5. a) If a fly landed on the outer edge of a $45\ rpm$ record, how far would it travel in 1 minute?

   b) How far if it was perched on a $33\frac{1}{3}\ rpm$ record?

   c) How far if it was perched on a $78\ rpm$ record?

SET

Topic: Using trigonometric ratios to solve problems

Perhaps you have seen *The London Eye* in the background of a recent James Bond movie or on a television show. When it opened in March of 2000, it was the tallest Ferris wheel in the world. The passenger capsule at the very top is 135 meters above the ground. The diameter is 120 meters.

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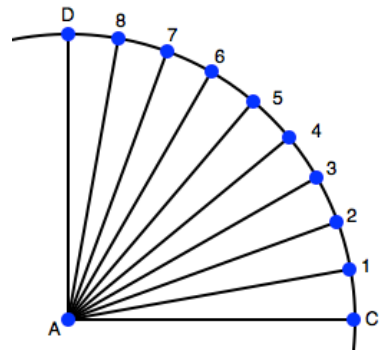
6. How high is the center of the Ferris wheel?
7. How far from the ground is the very bottom passenger capsule?
8. Assume there are 36 passenger capsules, evenly spaced around the circumference. Find the height from the ground of each of the *numbered* passenger capsules shown in the figure. (Use the figure at the right to help you think about the problem.)



GO

Topic: Connecting the trigonometric ratios

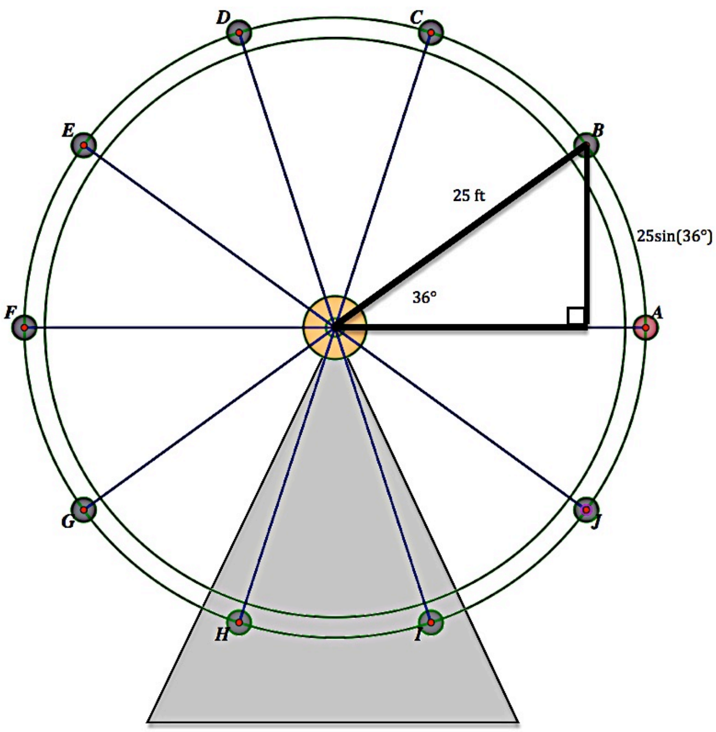
Find the other two trig ratios based on the one that is given.

9. $\sin\theta = \frac{4}{5}$	$\cos\theta =$	$\tan\theta =$
10. $\sin\theta =$	$\cos\theta = \frac{5}{13}$	$\tan\theta =$
11. $\sin\theta =$	$\cos\theta =$	$\tan\theta = 1$
12. $\sin\theta = \frac{1}{2}$	$\cos\theta =$	$\tan\theta =$
13. $\sin\theta =$	$\cos\theta = \frac{9}{41}$	$\tan\theta =$
14. $\sin\theta =$	$\cos\theta =$	$\tan\theta = \sqrt{3}$

6.2 "Sine" Language

A Solidify Understanding Task

In the previous task, *George W. Ferris' Day Off*, you probably found Carlos' height at different positions on the Ferris wheel using right triangles, as illustrated in the following diagram.



Recall the following facts from the previous task:

- The Ferris wheel has a radius of 25 feet
- The center of the Ferris wheel is 30 feet above the ground

Carlos has also been carefully timing the rotation of the wheel and has observed the following additional fact:

- The Ferris wheel makes one complete revolution counterclockwise every 20 seconds

1. How high will Carlos be 2 seconds after passing position A on the diagram?

2. Calculate the height of a rider at each of the following times t, where t represents the number of seconds since the rider passed position A on the diagram. Keep track of any regularities you notice in the ways you calculate the height. As you calculate each height, plot the position on the diagram.

Elapsed time since passing position A	Calculations	Height of the rider
1 sec		
2 sec		
2.5 sec		
3 sec		
6 sec		
8 sec		
14.5 sec		
18 sec		
23 sec		
28 sec		
36 sec		
37 sec		
40 sec		

3. Examine your calculations for finding the height of the rider during the first 5 seconds after passing position A (the first few values in the above table). During this time, the angle of rotation of the rider is somewhere between 0° and 90°. Write a general formula for finding the height of the rider during this time interval.

4. How might you find the height of the rider in other "quadrants" of the Ferris wheel, when the angle of rotation is greater than 90°?

6.2 "Sine" Language – Teacher Notes

A Solidify Understanding Task

Purpose: The purpose of this task is to extend the strategies used in the previous task for finding the height of a rider on a stationary Ferris wheel, to finding the height of a rider after an interval of time has elapsed since the rider passed the point farthest to the right of the wheel. This motion of the rider can be modeled by an angle of rotation drawn in "standard" position (i.e., with the initial ray pointing to the right and with a positive angle representing counterclockwise rotation). Students will identify that the function $height = 30 + 25\sin(18t)$ gives the height of a rider after t seconds, at least for $0 < t < 5$ seconds—times where $18t$ gives an angle between 0° and 90°, and therefore $\sin(18t)$ can be found using right triangle trigonometry. For $t > 5$ seconds, students will need to consider the related right triangles in each quadrant, and modify their formula so that the definition of sine as a ratio of sides of a right triangle holds. This leads to a piecewise-defined function for the height of the rider—a dilemma that will be resolved in future tasks when the definition of sine is extended.

Core Standards Focus:

F.TF.5 Choose trigonometric functions to model periodic phenomena with specified amplitude, frequency, and midline.★

Related Standards: G.SRT.8

Standards for Mathematical Practice:

SMP 4 – Model with mathematics

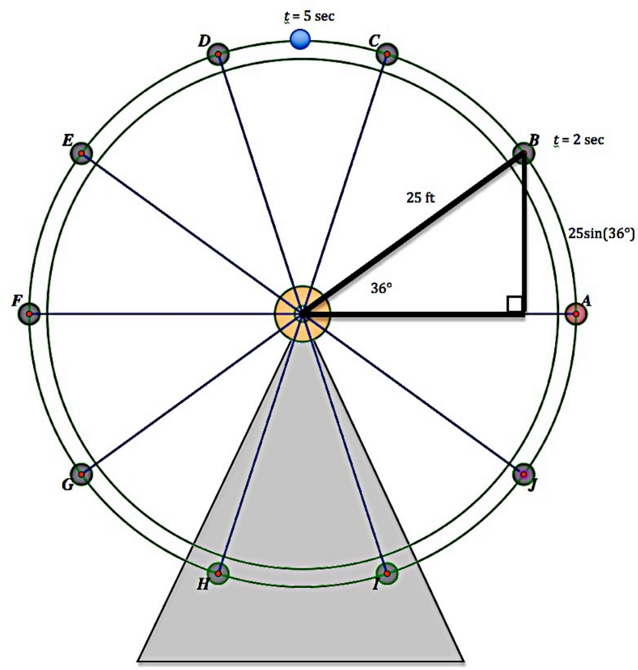
SMP 7 – Look for and make use of structure

Vocabulary: Students will generate a new quantity, *angular speed,* to find the angle of rotation as a function of elapsed time. The angular speed of the Ferris wheel is found by dividing 360° by the time it takes to make one complete revolution. The angular speed for the Ferris wheel in the task is 18° per second.

The Teaching Cycle:

Launch (Whole Class):

Begin by reviewing the work from the previous task by asking a student to describe how the triangle drawn on the diagram of the Ferris wheel (see the first page of the task) was used to determine the height of the associated point from the ground. Then, point out the new information—the wheel makes one complete revolution counterclockwise every 20 seconds. With this additional information, ask students to calculate Carlos' height 2 seconds after he passes point A. Give students a few minutes to work on this problem. Help students recognize that an important related fact is that the wheel rotates 18° per second. If students are finding it difficult to notice this, ask, "If it takes 20 seconds to make one complete revolution, how many seconds would it take to rotate to this position where the spoke is 36° from the horizontal?" Students might set up a proportion, $\frac{20\text{ sec}}{360°} = \frac{t\text{ sec}}{36°}$, or divide 360° by 20 to find the *angular speed* of 18°/sec. Alternatively, you might ask, "How many seconds would it take for the rider to move from position A to position B?"



Since they have already calculated this height in the previous task, students can record this height directly on their chart (see question 2). Tell students to watch for heights they have already calculated or related work they might use as they complete the table in question 2. Have students label the point at 36° as $t = 2$ seconds, $h = 44.7$ feet; then have them add and label the highest point on the Ferris wheel diagram as $t = 5$, $h = 55$ feet. Tell students they should plot and label other points on the Ferris wheel as they work on question 2.

Explore (Small Group):

Initially, students may need to think about the number of degrees of rotation associated with each time. They may reason that since the angular speed is 18° per second, Carlos will rotate 9° in a half-second or 54° in 3 seconds. Listen for students who recognize that they can use the expression *18t* inside the sine function. Also, watch for students who make use of related triangles to reduce the number of calculations they need to complete. For example, the work used to calculate the height at $t = 2$ seconds can be used to calculate the height at $t = 8$ seconds and $t = 18$ seconds due to the symmetry of the circle.

As students move into other "quadrants", such as when $t = 6$ seconds and the angle of rotation is 108°, they may calculate $\sin(108°)$ on their calculator without recognizing that a 108° angle doesn't make sense as an angle in right triangle trigonometry. Acknowledge that the calculator can do something we do not yet understand, and therefore we are not going to use these "mysterious, obtuse" values. Instead, ask them to do what they did in the previous task: draw a related right triangle—in this case a triangle with an acute angle measuring 72°—and use that triangle to calculate the height at $t = 6$ seconds or an angle of rotation of 108°. This is important work for developing understanding of trigonometric relationships, so don't skip over it by allowing students to use the calculator mindlessly. This dilemma will be resolved in future tasks when the definition of sine is extended.

Discuss (Whole Class):

Begin the discussion with question 3, where the angle of rotation is between 0° and 90° and t, the elapsed time, is between 0 and 5 seconds. Select a student who can present how we might generalize the computational work in this time interval using the formula $h(t) = 30 + 25\sin(18t)$.

Make sure all students understand the dilemma of using this formula for $t > 5$ since we have not established meaning for the sine of an angle greater than (or even equal to) 90°. If there are students who have responded to question 4, have them present their formulas for quadrants II, and have other students present their work for positions in quadrants III and IV. Even if their formulas are initially inaccurate, the work of resolving what to do in each quadrant is important work to discuss. This discussion should lead to the following piecewise-defined functions for one revolution of the wheel.

$$h(t)=\begin{cases} 30+25\sin(18t), & 0<t<5 \\ 30+25\sin(180-18t), & 5<t<10 \\ 30-25\sin(18t-180), & 10<t<15 \\ 30-25\sin(360-18t), & 15<t<20 \\ 55, & t=5 \\ 30, & t=0 \text{ or } t=10 \text{ or } t=20 \\ 5, & t=15 \end{cases}$$

Aligned Ready, Set, Go: Modeling Periodic Behavior 6.2

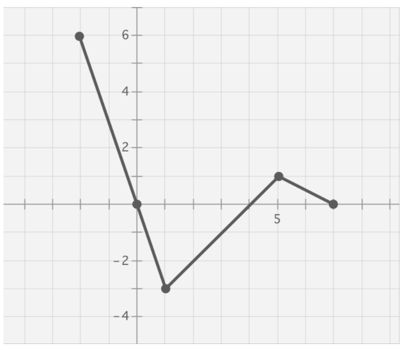
READY, SET, GO! Name Period Date

READY

Topic: Describing intervals from graphs

For each graph, write the interval(s) where $f(x)$ is positive and the interval(s) where it is negative.

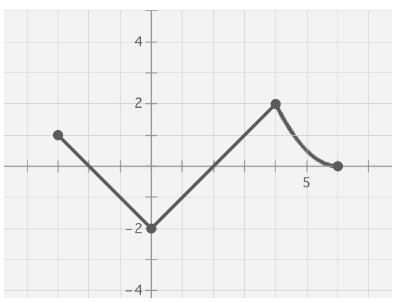
1.



Positive ______________________________

Negative ______________________________

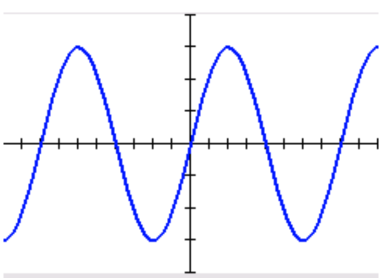
2.



Positive ______________________________

Negative ______________________________

3.

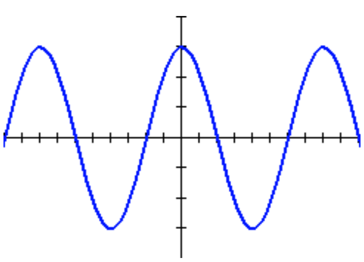


(The scale on the x-axis is in increments of 45°.)

Positive ______________________________

Negative ______________________________

4.



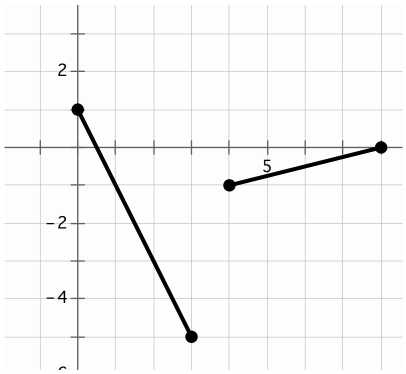
(The scale on the x-axis is in increments of 45°.)

Positive ______________________________

Negative ______________________________

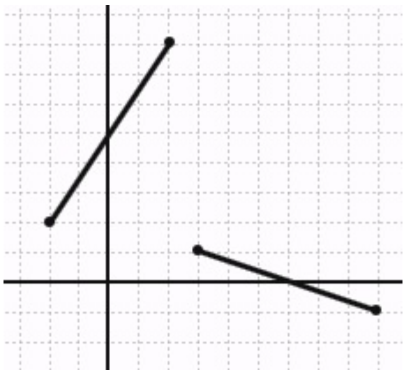
Write the piece-wise equations for the given graphs.

5.



Equation:

6.



Equation:

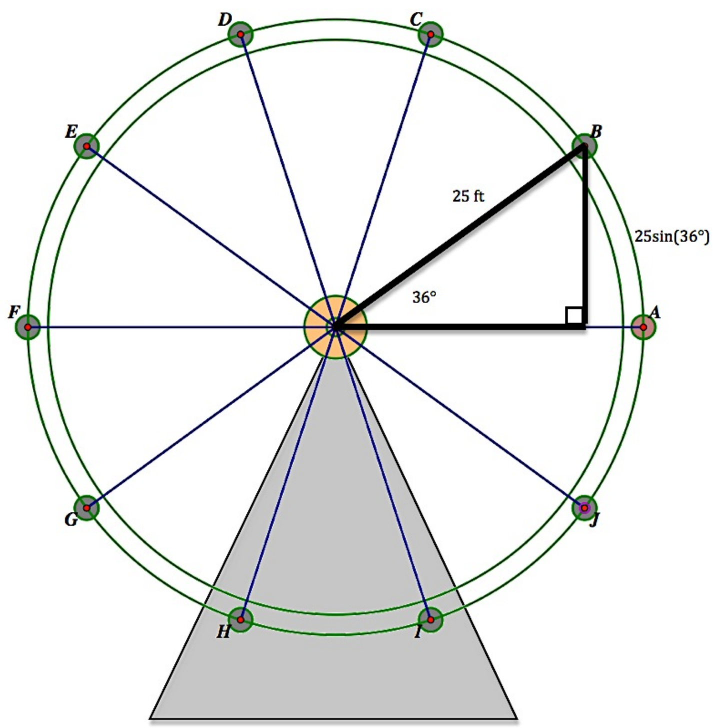
SET

Topic: Calculating sine as a function of time

Recall the following facts from the classroom task:

- The Ferris wheel has a radius of 25 feet
- The center of the Ferris wheel is 30 feet above the ground

Due to a safety concern, the management of the amusement park decides to slow the rotation of the Ferris wheel from 20 seconds for a full rotation to **30 seconds for a full rotation**.



7. Calculate how high a rider will now be 2 seconds after passing position A on the diagram.

8. Calculate the height of a rider at each of the following times t, where t represents the number of seconds since the rider passed position A on the diagram. As you calculate each height, plot the position on the diagram. Connect the center of the circle to the point you plotted. Then draw a vertical line from the plotted point on the Ferris wheel to the line segment AF in the diagram. Each time you should get a right triangle similar to the one in the figure.

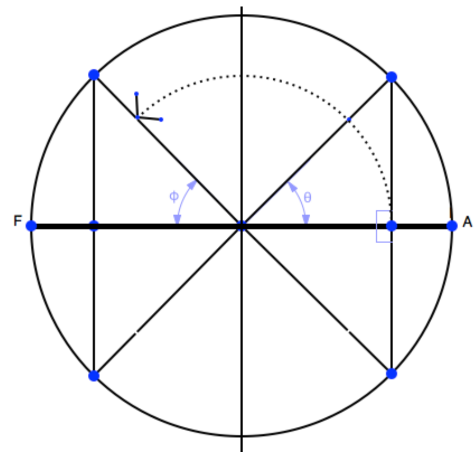
Elapsed time since passing position A	**Calculations**	**Height of the rider (in feet)**
1 sec		
3 sec		
5 sec		
7 sec		
8 sec		
11 sec		
14 sec		
15 sec		
16 sec		
20 sec		
22 sec		
23 sec		
25 sec		
27 sec		
30 sec		

9. How did the position of the triangles you drew change between 7 seconds and 8 seconds?

10. How did the triangles you drew change between 14, 15, and 16 seconds?

11. How did the triangles you drew change between 22 seconds and 23 seconds?

12. Describe a relationship between the orientation of the right triangles around the circle and the angle of rotation. Use the diagram to help you think about the question. (The dotted arc shows the angle of rotation.)



GO

Topic: Finding missing angles in triangles

Find the measure of each acute angle of right triangle ABC with $m\angle C = 90°$.

Round your answers to the nearest degree.

13. $a = 3\ in\ \ c = 5\ in$

14. $a = 5\ ft\ \ c = 10\ ft$

15. $a = 9.1\ cm\ \ c = 12.3\ cm$

16. $a = 14.1\ cm\ \ c = 18.\ cm$

17. $a = 9.7\ in\ \ b = 12.7\ in$

18. $a = 14.6\ ft\ \ c = 20.3\ ft$

6.3 More "Sine" Language

A Solidify Understanding Task

Clarita is helping Carlos calculate his height at different locations around a Ferris wheel. They have noticed when they use their formula $h(t) = 30 + 25\sin(\theta)$ their calculator gives them correct answers for the height even when the angle of rotation is greater than 90°. They don't understand why, since right triangle trigonometry only defines the sine for acute angles.

Carlos and Clarita are making notes of what they have observed about this new way of defining the sine that seems to be programmed into the calculator.

Carlos: "For some angles the calculator gives me positive values for the sine of the angle, and for some angles it gives me negative values."

1. Without using your calculator, list at least five angles of rotation for which the value of the sine produced by the calculator should be positive.

2. Without using your calculator, list at least five angles of rotation for which the value of the sine produced by the calculator should be negative.

Clarita: "Yeah, and sometimes we can't even draw a triangle at certain positions on the Ferris wheel, but the calculator still gives us values for the sine at those angles of rotation."

3. List possible angles of rotation that Clarita is talking about—positions for which you can't draw a reference triangle. Then, without using your calculator, give the value of the sine that the calculator should provide at those positions.

Carlos: "And, because of the symmetry of the circle, some angles of rotation should have the same values for the sine."

4. Without using your calculator, list at least five pairs of angles that should have the same sine value.

Clarita: "Right! And if we go around the circle more than once, the calculator still gives us values for the sine of the angle of rotation, and multiple angles have the same value of the sine."

5. Without using your calculator, list at least five sets of multiple angles of rotation where the calculator should produce the same value of the sine.

Carlos: "So how big can the angle of rotation be and still have a sine value?"

Clarita: "Or how small?"

6. How would you answer Carlos and Clarita's questions?

Carlos: "And while we are asking questions, I'm wondering how big or how small the value of the sine can be as the angles of rotation get larger and larger?"

7. Without using a calculator, what would your answer be to Carlos' question?

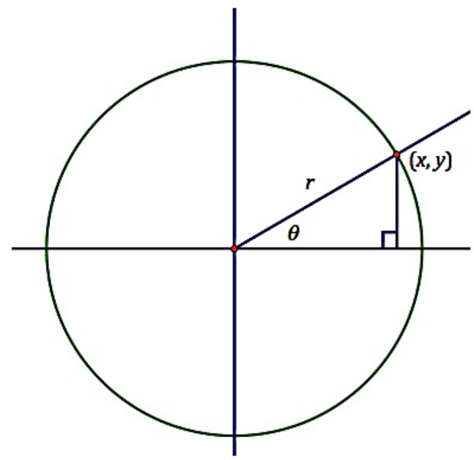
Clarita: "Well, whatever the calculator is doing, at least it's consistent with our right triangle definition of sine as the ratio of the length of the side opposite to the length of the hypotenuse for angles of rotation between 0 and 90°."

Part 2

Carlos and Clarita decide to ask their math teacher how mathematicians have defined sine for angles of rotation, since the ratio definition no longer holds when the angle isn't part of a right triangle. Here is a summary of that discussion.

We begin with a circle of radius *r* whose center is located at the origin on a rectangular coordinate grid. We represent **an angle of rotation in standard position** by placing its vertex at the origin, the *initial ray* oriented along the positive *x*-axis, and its *terminal ray* rotated θ degrees counterclockwise around the origin when θ is positive and clockwise when θ is negative. Let the ordered pair (x, y) represent the point when the terminal ray intersects the circle. (See the diagram at right, which Clarita diligently copied into her notebook.)

In this diagram, angle θ is between 0 and 90°; therefore, the terminal ray is in quadrant I. A right triangle has been drawn in quadrant I, similar to the right triangles we have drawn in the Ferris wheel tasks.

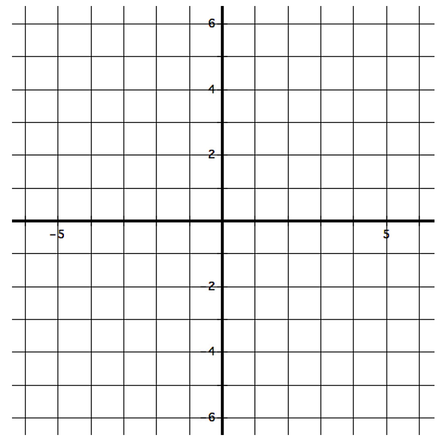


8. Based on this diagram and the right triangle definition of the sine ratio, find an expression for $sin\ \theta$ in terms of the variables *x, y* and *r.*

$sin\ \theta =$

We will use this definition for <u>any</u> angle of rotation. Let's try it out for a specific point on a particular circle.

9. Consider the point (-3, 4), which is on the circle $x^2 + y^2 = 25$.



   a. What is the radius of this circle?

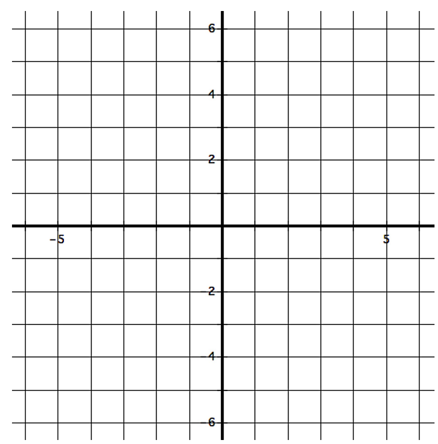
   b. Draw the circle and the angle of rotation, showing the initial and terminal ray.

   c. For the angle of rotation you just drew, what would the value of the sine be if we use the definition we wrote for sine in question 8?

   d. What is the measure of the angle of rotation? How did you determine the size of the angle of rotation?

   e. Is the calculated value based on this definition the same as the value given by the calculator for this angle of rotation?

10. Consider the point (-1, -3), which is on the circle $x^2 + y^2 = 10$.



   a. What is the radius of this circle?

   b. Draw the circle and the angle of rotation, showing the initial and terminal ray.

   c. For the angle of rotation you just drew, what would the value of the sine be if we use the definition we wrote for sine in question 8?

   d. What is the measure of the angle of rotation? How did you determine the size of the angle of rotation?

   e. Is the calculated value based on this definition the same as the value given by the calculator for this angle of rotation?

6.3 More "Sine" Language – Teacher Notes

A Solidify Understanding Task

Purpose: In the previous task we had to write multiple expressions to represent Carlos' height on a Ferris wheel, depending upon which quadrant of the wheel he was located in. Our definition of sine needs to be modified if we want to use a single expression to represent Carlos' height at every point on the wheel. The purpose of this task is to introduce a new definition of the sine function, based on an angle of rotation approach. This is the first extension of the definition of trigonometry—to define sine (and later, cosine and tangent) so the definitions apply to angles of rotation, and not just to the acute angles of right triangles.

Teacher's Note: Right triangle trigonometry is based on AA-similarity: if two right triangles also contain a pair of congruent corresponding acute angles, then the right triangles are similar. This means ratios of corresponding sides are proportional. These ratios are given names: *sine* is the ratio of the length of the side opposite of the acute angle and the length of the hypotenuse, *cosine* is the ratio of the length of the side adjacent to the acute angle and the length of the hypotenuse, and *tangent* is the ratio of the length of the opposite side to the length of the adjacent side. Using these definitions, and a table of trig values, one can determine missing lengths and angles in right triangles.

Right triangle trigonometry can be extended to define trigonometric functions for angles of rotation. To do so, a standard position for an angle of rotation is defined: locate the vertex of the angle of rotation at the origin of a Cartesian coordinate system, with the initial ray pointed along the positive *x*-axis. Counterclockwise angles of rotation are considered positive angles; clockwise rotations are considered negative angles. Angles of rotation can exceed 360°, indicating the amount of rotation has exceeded one complete turn. To define sine, cosine and tangent for an angle of rotation θ, select a point (x, y) on the terminal ray and calculate its distance *r* from the origin using $r = \sqrt{x^2 + y^2}$. The sine of the angle of rotation is defined by the ratio $^y/_r$, cosine by the ratio $^x/_r$, and tangent by the ratio $y/_x$. In the first quadrant, for angles between 0° and 90°, these ratios are consistent with the right triangle definitions of the trig ratios, since *x* measures the length of the adjacent side of a right triangle drawn in quadrant I, *y* measures the length of the side opposite, and *r* measures the length of the hypotenuse. This new definition, however, allows one to define trig values for angles that measure more than 90° or less than 0°. If the point on the terminal ray is chosen to be the point that lies 1 unit away from the origin (that is, *r* = 1) then the point lies on the unit circle, and the coordinates of the points on the unit circle are given by $(\cos\theta, \sin\theta)$.

With this new definition of the trigonometric functions, trigonometry can be applied to periodic behavior. The simplest application is the periodic behavior of an object moving around a circular path. More complex applications would include those where the behavior is periodic or cyclical, but not circular. Functions of the form $y = a\sin(bt)$ or $y = a\cos(bt)$ keep track of the *x*- and *y*-coordinates of a point moving around a circle of radius a with an angular speed of b degrees per second, which is equivalent to saying the period is $\frac{360°}{b}$ seconds.

Core Standards Focus:
F.TF.2 Explain how the unit circle in the coordinate plane enables the extension of trigonometric functions to all real numbers, interpreted as radian measures of angles traversed counterclockwise around the unit circle. (**Note:** *This is the first in a series of tasks that will extend trigonometric functions to all real numbers as inputs. In this task, the input into the trigonometric function represents an angle of rotation measured in degrees.*)

Standards for Mathematical Practice:
SMP 8 – Look for and express regularity in repeated reasoning
Vocabulary: Students will be introduced to *angles of rotation in standard position* as angles whose vertex is located at the origin, with its *initial ray* oriented along the positive *x*-axis, and its *terminal ray* rotated θ degrees counterclockwise around the origin when θ is positive and clockwise when θ is negative.

The Teaching Cycle:

Launch (Whole Class):

Remind students of the dilemma of having to write different functions for the height of a rider on a Ferris wheel if we can only use angles between 0 and 90° for which the sine ratio is defined. Examine what happens on a calculator when the function *h(t) =30 + 25sin(18t)* is used for values of *t* where θ = 18*t* is greater than 90°. Once it is observed that the calculator gives correct results, regardless of the location of the point on the wheel, turn to the introduction of the task and have students work with a partner to answer questions 1-7. Point out that these questions are based on Carlos and Clarita's observations about what is unique and important to notice about the definition of sine that is programmed into the calculator.

After a few minutes, discuss each of these questions before moving to the next part of the task, where students are given a definition of the sine function based on an angle of rotation approach, and asked to apply that definition in a couple of specific cases. As you discuss Carlos and Clarita's observations and questions, make sure the following issues get addressed and justified:

- $\sin(-\theta) = -\sin(\theta)$
- $\theta + 2\pi n$ or $\theta - 2\pi n$ yield a series of co-terminal angles that have the same sine value

Explore (Small Group):

Prepare students for part 2 of the task by introducing the diagram of a circle intersected by an initial and a terminal ray. Read through the definition of drawing an angle of rotation in standard position and label the initial and terminal rays on this diagram. Then have students work on questions 8-10.

If students are having difficulty on question 8, ask them how the coordinates of the point (*x, y*) relate to the lengths of the sides of the right triangle. Once the sides have been labeled with lengths *x* and *y* students should have no problem writing $\sin(\theta) = \frac{y}{r}$.

Students may have difficulty finding the angles of rotation asked for in questions 9d and 10d. Remind them that they have used the inverse sine function to find the measure of angles whose trigonometric values are known. However, in each of these cases, a calculator will give them an angle measure that is clearly not the correct angle of rotation. (Make sure students have set their calculators in degree mode.) Ask students why the calculated value may be incorrect and how they might use the value produced by the calculator to find the correct measure of the angle of rotation. Listen for students who can explain this idea during the whole class discussion.

Discuss (Whole Class):

Make sure that students have agreed on the new definition of the sine by having a student present the answer to question 8. Then focus the discussion on questions 9 and 10 by having different students draw and label the initial and terminal rays for each angle of rotation (parts a and b). Once correct drawings are on the board, ask students to estimate the size of the angle of rotation. For question 9, students should recognize that the angle should be between 90° and 180° since the terminal ray is in the second quadrant, and that the angle of rotation for question 10 should be between 180° and 270°. Have students share how they found the actual angles of rotation using the inverse sine function on the calculator and then interpreting the information provided by the calculator. It may be necessary to draw the terminal ray in the position suggested by the calculator, and see how this angle relates to the actual angle of rotation. Once students have agreed on the measure of the angle of rotation, compare the sine value of that angle as calculated using both the definition $\sin(\theta) = \frac{y}{r}$ as well as evaluating $\sin(\theta)$ on a calculator to show that the values are in agreement.

Aligned Ready, Set, Go: Modeling Periodic Behavior 6.3

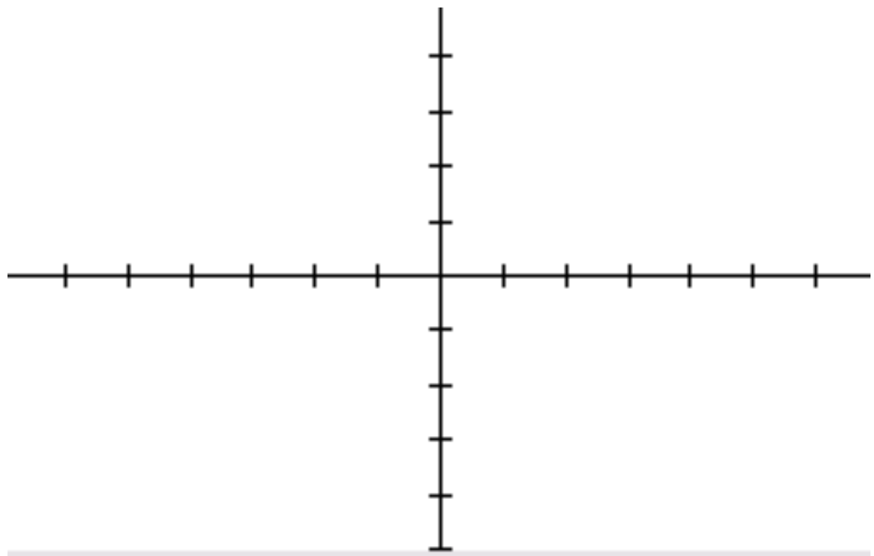
READY, SET, GO! Name Period Date

READY

Topic: Graphing a curve

1. Graph the table of values. Connect your points with a smooth curve.

x	*y*
-6	0
-5	-3
-4	-4
-3	-3
-2	0
-1	3
0	4
1	3
2	0
3	-3
4	-4
5	-3
6	0



2. Identify the maximum and minimum values of the curve.

3. This curve repeats itself. (It's called a *periodic function*.) Find the length of the interval that would allow you to see **exactly** one full length of the curve.

4. The curve is positive on the interval (-2, 2). Identify two more intervals where this curve will be positive.

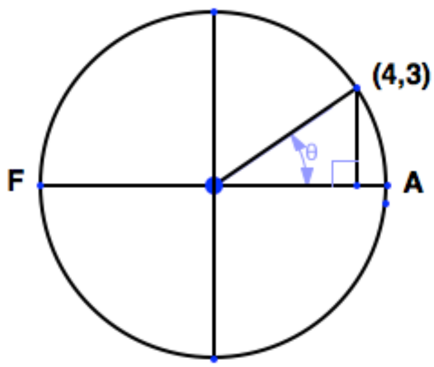
SET

Topic: Finding values of sine in the coordinate plane

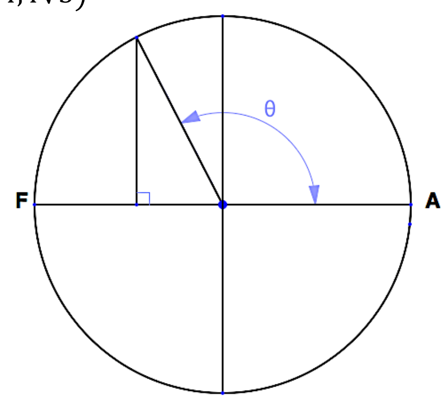
Use the given point on the circle to find the value of $sin\theta$. Then find the value of θ.

Recall $r = \sqrt{x^2 + y^2} and\ sin\theta = \frac{y}{r}.$

5.



6. $(-4, 4\sqrt{3})$



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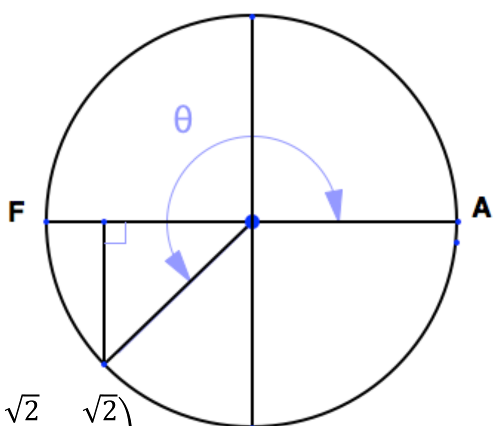
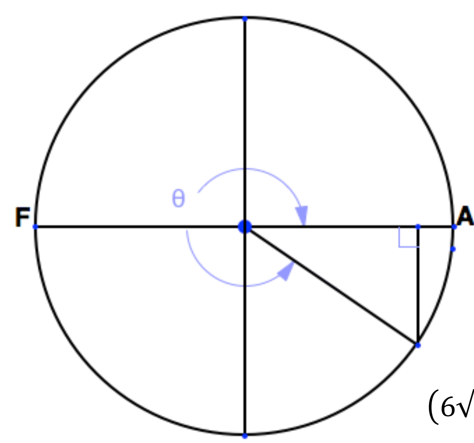


$\left(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$

8.



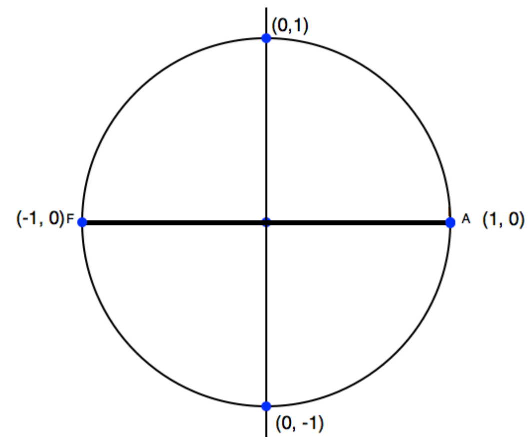
$(6\sqrt{3}, -6)$

9. In each graph above, the angle of rotation is indicated by an arc and θ. Describe the angles of rotation that make the y-values of the points be positive and the angles of rotation that make the y-values be negative.

10. What do you notice about the y-values and the value of sine in the graphs above?

11. In the graph at the right, the radius of the circle is 1 unit. The intersections of the circle and the axes are labeled. Based on your observation in #10, what do you think the value of sine might be for the following values of θ:

90°? 180°? 270°? 360°?



GO

Topic: Solving problems using right angle trigonometry

Make a sketch of the following problems, then solve.

12. A kite is aloft at the end of a 1500 foot string. The string makes an angle of 43° with the ground. How far above the ground is the kite? (Round your answer to the nearest foot.)

13. A ladder leans against a building. The top of the ladder reaches a point on the building that is 12 feet above the ground. The foot of the ladder is 4 feet from the building. Find to the nearest degree the measure of the angle that the ladder makes with the level ground. What is the angle the ladder makes with the building?

14. The shadow of a flagpole is 40.6 meters long when the angle of elevation of the sun is 34.6°. Find the height of the flagpole.

15. The angle of depression from the top of a building to a car parked in the parking lot is 32.5°. How far from the top of the building is the car on the ground, if the building is 252 meters high?

6.4 More Ferris Wheels

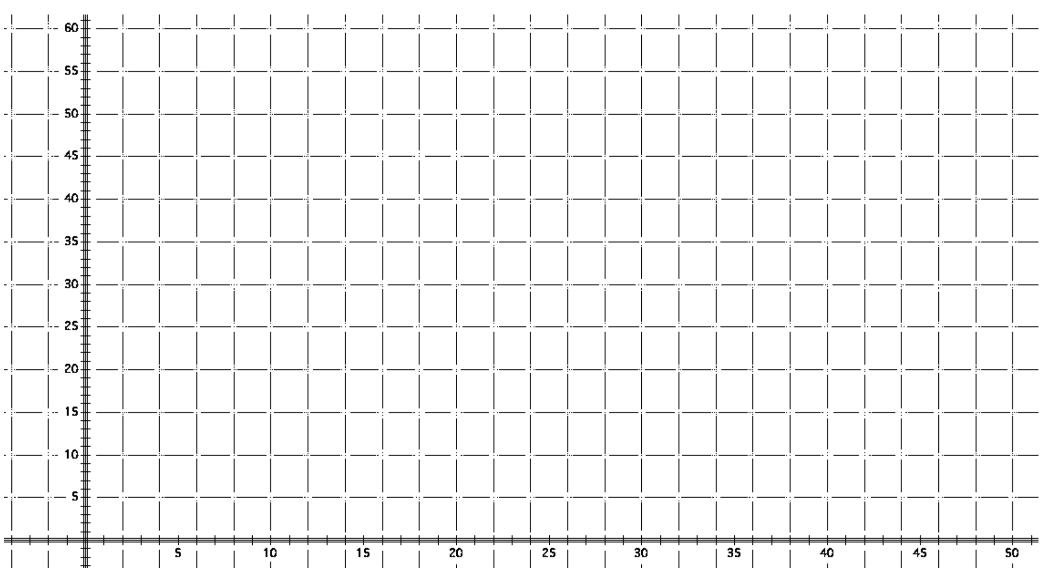
A Solidify Understanding Task

In a previous task, *"Sine" Language*, you calculated the height of a rider on a Ferris wheel at different times *t*, where *t* represented the elapsed time after the rider passed the position farthest to the right of the Ferris wheel.

Recall the following facts for the Ferris wheel in the previous tasks:

- The Ferris wheel has a radius of 25 feet
- The center of the Ferris wheel is 30 feet above the ground
- The wheel makes one complete revolution counterclockwise every 20 seconds

1. Based on the data you calculated, as well as any additional insights you might have about riding on Ferris wheels, sketch a graph of the height of a rider on this Ferris wheel as a function of the time elapsed since the rider passed the position farthest to the right of the Ferris wheel. (We can consider this position as the rider's starting position at time $t = 0$.)

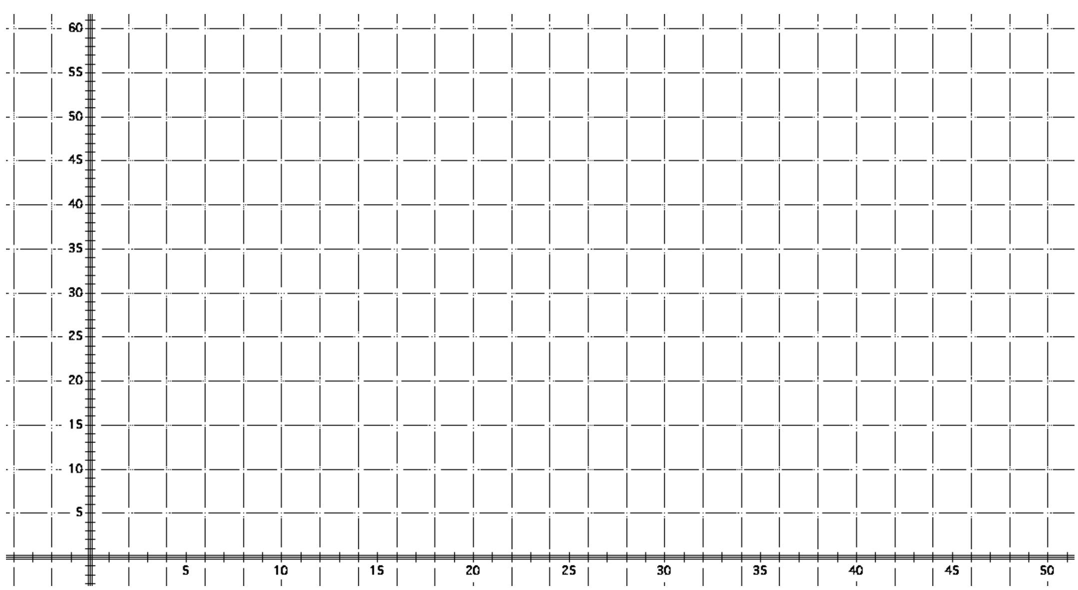


2. Write the equation of the graph you sketched in question 1.

3. Of course, Ferris wheels do not all have this same radius, center height, or time of rotation. Describe a different Ferris wheel by changing some of the facts listed above. For example, you can change the radius of the wheel, the height of the center, or the amount of time it takes to complete one revolution. You can even change the direction of rotation from counterclockwise to clockwise. If you want, you can change more than one fact. Just make sure your description seems reasonable for the motion of a Ferris wheel.

   Description of my Ferris wheel:

4. Sketch a graph of the height of a rider on your Ferris wheel as a function of the time elapsed since the rider passed the position farthest to the right of the Ferris wheel.



5. Write the equation of the graph you sketched in question 4.

6. We began this task by considering the graph of the height of a rider on a Ferris wheel with a radius of 25 feet and center 30 feet off the ground, which makes one revolution counterclockwise every 20 seconds. How would your graph change if:
   - the radius of the wheel was larger or smaller?
   - the height of the center of the wheel was greater or smaller?
   - the wheel rotates faster or slower?

7. How does the equation of the rider's height change if:
   - the radius of the wheel is larger or smaller?
   - the height of the center of the wheel is greater or smaller?
   - the wheel rotates faster or slower?

8. Write the equation of the height of a rider on each of the following Ferris wheels t seconds after the rider passes the farthest right position.
   a. The radius of the wheel is 30 feet, the center of the wheel is 45 feet above the ground, and the angular speed of the wheel is 15 degrees per second counterclockwise

   b. The radius of the wheel is 50 feet, the center of the wheel is at ground level (you spend half of your time below ground), and the wheel makes one revolution *clockwise* every 15 seconds.

6.4 More Ferris Wheels – Teacher Notes

A Solidify Understanding Task

Purpose: The purpose of this task is to examine how changing the parameters in a function of the form $h(t) = a\sin(bt) + d$ affects the corresponding graph of the function. Students will make connections between the parameters in the equation, the description of the motion of the Ferris wheel, and the amplitude, period and midline of the graph. The *midline*, which lies halfway between the maximum and minimum points of the graph, depends upon the height of the center of the Ferris wheel and is represented by the value of the parameter d. The *amplitude*, or distance from the midline to the maximum and minimum points of the graph, depends upon the radius of the Ferris wheel and is represented by the value of the parameter a. The *period*, or interval before the graph repeats itself, depends upon the length of time of one complete revolution of the wheel. The parameter b represents the angular speed of the wheel—given in terms of degrees per second for this scenario—and is found by dividing 360° by the amount of time it takes to make one complete revolution. Students will observe that the graph is periodic—the rider returns to the same height every 20 seconds. Students should also note the characteristic shape of a sinusoidal graph—the smooth turns rather than sharp points at the maximum and minimum values, for example—and be able to justify the constantly changing slope of the graph in terms of the horizontal and vertical components of motion one experiences when riding on a Ferris wheel.

Core Standards Focus:

F.TF.5 Choose trigonometric functions to model periodic phenomena with specified amplitude, frequency, and midline.★

F.IF.4 For a function that models a relationship between two quantities, interpret key features of graphs and tables in terms of the quantities, and sketch graphs showing key features given a verbal description of the relationship. Key features include: intercepts; intervals where the function is increasing, decreasing, positive, or negative; relative maximums and minimums; symmetries; end behavior; and periodicity.★

F.BF.3 Identify the effect on the graph of replacing f(x) by f(x) + k, k f(x), f(kx), and f(x + k) for specific values of k (both positive and negative); find the value of k given the graphs. Experiment with cases and illustrate an explanation of the effects on the graph using technology. Include recognizing even and odd functions from their graphs and algebraic expressions for them.

Instructional Note for Secondary Math III: *For F.BF.3, note the effect of multiple transformations on a single function and the common effect of each transformation across function types. Include functions defined only by a graph.*

Related Standards: F.BF.1c

Standards for Mathematical Practice:
SMP 3 – Construct viable arguments and critique the reasoning of others
SMP 7 – Look for and make use of structure

Vocabulary: Students will be introduced to the characteristic features of sinusoidal graphs—*amplitude*, *period* and *midline*—and the *parameters* in the equation of the trigonometric function that determine these features.

The Teaching Cycle:
Launch (Whole Class):
Students may need some help with the descriptive language of the function given in this task: *the height of a rider on this Ferris wheel as a function of the time elapsed since the rider passed the position farthest to the right on the Ferris wheel*. Point out that these words mean that at $t = 0$, the rider is *farthest to the right* and moving up and to the left on the wheel. This implies that the y-intercept on our graph will be the same as the height of the center of the wheel. After clarifying the

language and the implied motion of the rider, set students to work graphing the height of the rider as a function of elapsed time.

Explore (Small Group):

Watch for different strategies students might use to sketch this graph. Since we have calculated the heights at twelve different positions in a previous task—the endpoints of the ten spokes, plus the highest and lowest points on the wheel—students may plot that data at appropriate points in time, and then connect those points with some kind of curve. Others may just plot the points where the rider is the same height as the center, as well as the highest and lowest points, and then connect them with some kind of curve. Students who plot fewer points often connect those points with straight line segments. Students who plot more points begin to notice the characteristic curve of the sinusoidal graph. Listen for students who may relate the shape of the graph to the horizontal and vertical components of the motion of the rider. For example, they may say that the behavior of the graph at the maximum values should be a smooth turn, rather than coming to a sharp point, since the rider is moving almost horizontally there, with very little up or down motion. Students might note that the rider's height increases more during the first 2-second interval after passing the farthest right position than in the 2nd 2-second interval, and that the rider splits the 3rd 2-second interval going up and then down. These are important insights and need to be highlighted in the whole class discussion to make sense of the shape of the graph between plotted data points.

For question 2, students may wonder if they need to write the piecewise-defined function for the height of the rider that they wrote in task 5.2. Ask them what the implications are for the extended definition of the sine given in task 5.3. They should be able to argue that we have defined $\sin\theta$ for all angles of rotation in a way that allows us to write a single expression for the height of the rider: $h(t) = 25\sin(18t) + 30$. Press students to verify that this function does produce correct values of the height of the rider for all values of t. Students should particularly note that sometimes $\sin(18t)$ will be negative—specifically for values of t that locate the rider in the 3rd and 4th quadrants of the wheel—and therefore, we will get values of the height that are correctly located below the center of the wheel.

If it feels like students need clarification of these ideas before moving on with the rest of the task, stop and have a brief whole class discussion. The goal of this discussion is to get out on the table for all students ideas about the shape of the graph and about using a single expression for the rider's height.

As the exploration continues, listen for students who are connecting the changes they make in the description of the Ferris wheel to their equations and graphs. If students are having difficulty keeping track of more than one parameter change at a time, suggest they only change one fact in their description of the Ferris wheel and think about how that one change would affect the graph. Select students who can discuss the effects of changing a single fact—radius, center height, amount of time for one revolution—in both the graph and the equation to present first, followed by students who can describe the effect of changing multiple facts.

Discuss (Whole Class):
Start by examining the effects of a single parameter change on the graph and equation, perhaps by suggesting a particular change such as, "What would happen if the radius of the wheel was only 20 feet?" Have a presenter superimpose this graph on top of the graph of the original Ferris wheel. Use words like *midline* and *amplitude* to describe the features of the graphs and to note which features change.

Once students have discussed how changing the radius or height of the center affects the graph and the equation, move to a discussion of the *period*. Unlike the radius and center height, which are represented directly by parameters in the expression for the height of the rider, the period does not show up explicitly in the expression. Instead, the angular speed—which we have to calculate from the period—shows up as a parameter in the expression. (If students are familiar with function transformation concepts, this would be a good time to note that doubling the angular speed compresses the horizontal shape of the graph, whereas doubling the radius stretches the vertical shape of the graph. There are some powerful connections that can be made to observations students have made previously with other functions, in terms of function transformations. Allow these ideas to surface naturally in the discussion and pursue connections that students might identify.) Question 8 gets at the relationship between period and angular speed by giving students the angular speed in one context and the period in another. If students have not had time to finish question 8, assign it as homework and discuss it during the following class period.

Have at least one student present who changed several facts simultaneously to point out that each parameter change is independent of the others, and they can think of the transformations to the graph sequentially: find the period, find the amplitude, find the midline; or in terms of transformations: horizontally stretch or compress the graph to fit the period, vertically stretch or compress the graph to fit the radius, shift the graph up or down to fit the center.

If available, have a student who changed the direction of rotation to clockwise present last. Students should note that since the graph is reflected over the midline, the term $25\sin(18t)$ must be made negative so that we subtract a distance from the height of the center during the interval $0 < t < 5$, rather than adding. If no one made this change in the descriptions of their Ferris wheels, the issue will still come up in question 8b. Make sure that question 8 gets discussed, either as part of today's discussion or at the beginning of the next class period.

Aligned Ready, Set, Go: Modeling Periodic Behavior 6.4

READY, SET, GO! Name Period Date

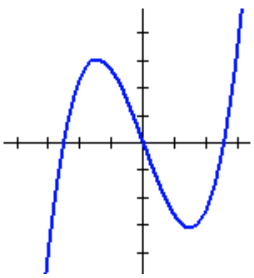
READY

Topic: Identifying *even* and *odd* functions from a graph

The *graphs* of even and odd functions make it easy to identify the type of function. Remember that an **even** function has a line of symmetry along the y-axis, while an **odd** function has 180° rotational symmetry.

Label the following functions as *even, odd*, or *neither.*

1.



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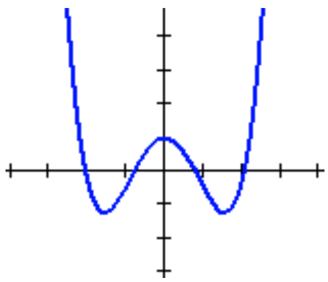

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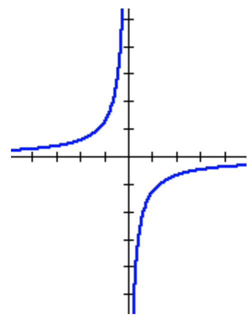

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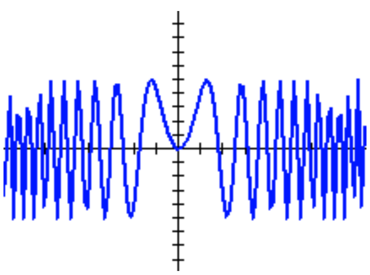

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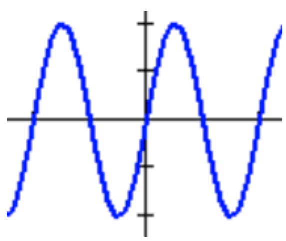

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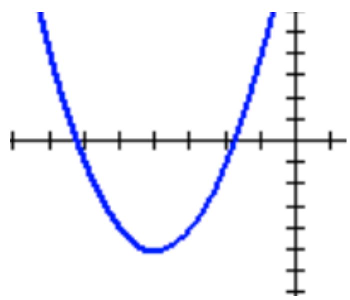

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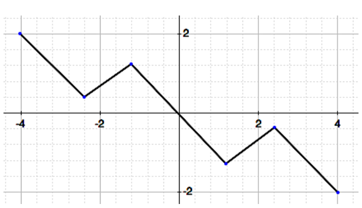

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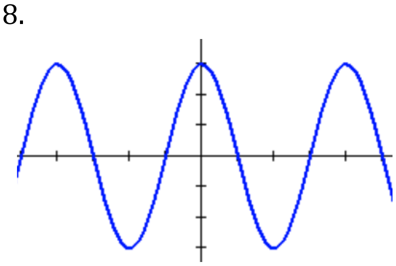

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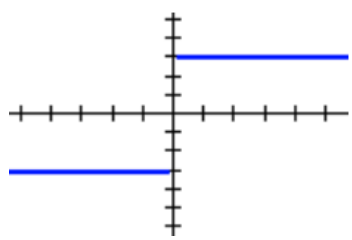

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SET

Topic: Describing transformations on functions

Describe the transformation(s) on the parabola in the following equations.

10. $y = x^2 + 5$ 11. $y = x^2 - 1$ 12. $y = -x^2$ 13. $y = 4x^2$

Match the equation with the correct graph. The scale of the x-axis is 90°. The scale of the y-axis is 1.

a. $y = \sin 2x$ b. $y = (\sin x) + 2$ c. $y = 3\sin x$

d. $y = -(\sin x) - 2$ e. $y = -2\sin x$ f. $y = 3\sin 2x$

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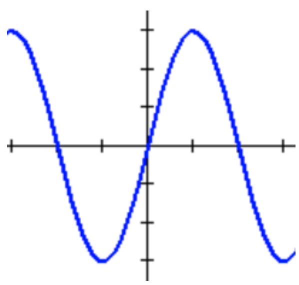

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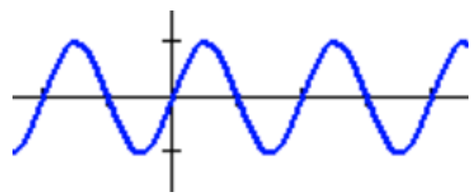

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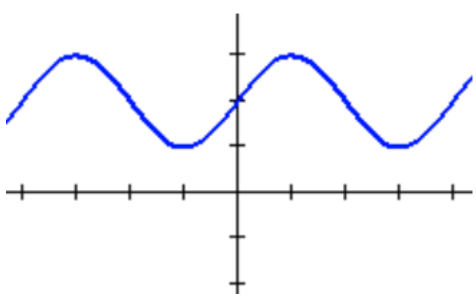

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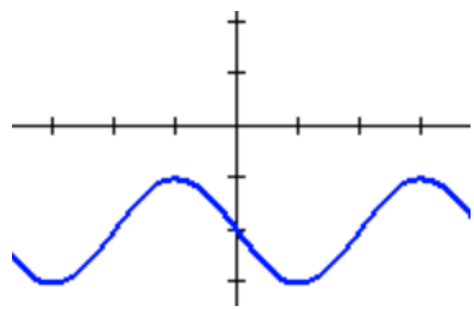

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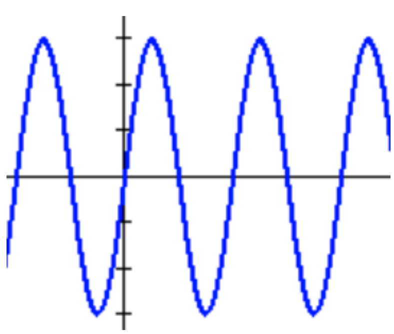

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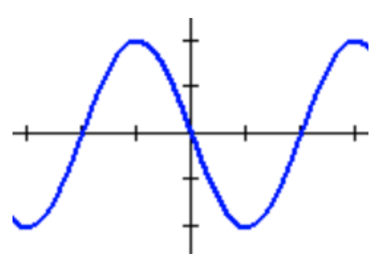


GO

Topic: Recognizing positive and negative angles of rotation

A positive angle of rotation is counter-clockwise. Let's find out why. In the following examples, indicate whether the customary direction of rotation is **counter-clockwise** by placing a (+) sign next to it or **clockwise** by placing a (–) sign next to it.

20. __________ Sprinters racing around a track

21. _________ The direction you turn the pages as you read a book

22. _________ A car in America traveling through a roundabout

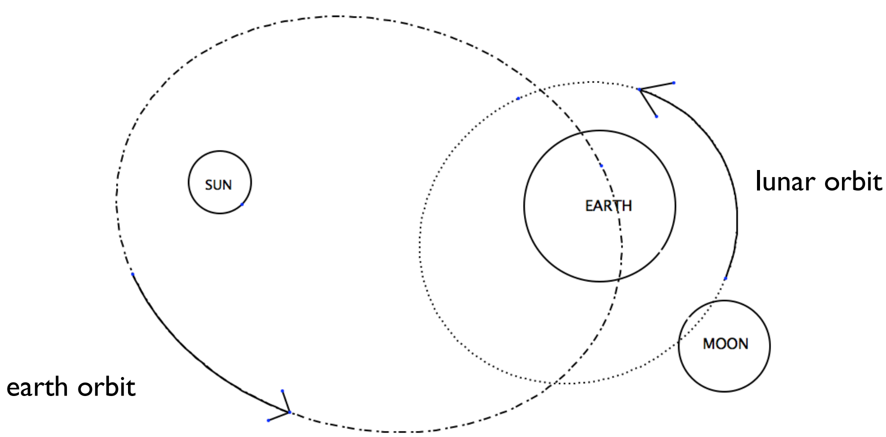
23. _________ Turning a water faucet to the on position

24. _________ A car in Australia circling in a roundabout (See sign.)

Wikimedia commons

25. ________ The rotation of the earth around the sun according to the diagram below.

26. ________ The rotation of the moon around the earth. (See diagram)



6.5 Moving Shadows

A Practice Understanding Task

In spite of his nervousness, Carlos enjoys his first ride on the amusement park Ferris wheel. He does, however, spend much of his time with his eyes fixed on the ground below him. After a while, he becomes fascinated with the fact that since the sun is directly overhead, his shadow moves back and forth across the ground beneath him as he rides around on the Ferris wheel.

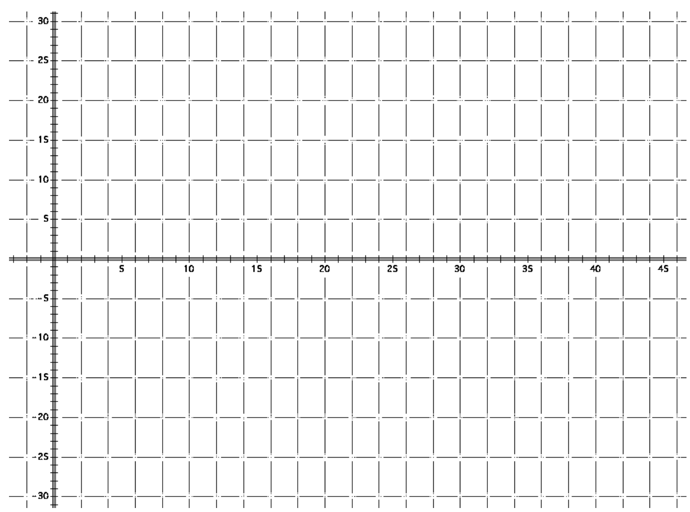
Recall the following facts for the Ferris wheel Carlos is riding:

- The Ferris wheel has a radius of 25 feet
- The center of the Ferris wheel is 30 feet above the ground
- The Ferris wheel makes one complete rotation counterclockwise every 20 seconds

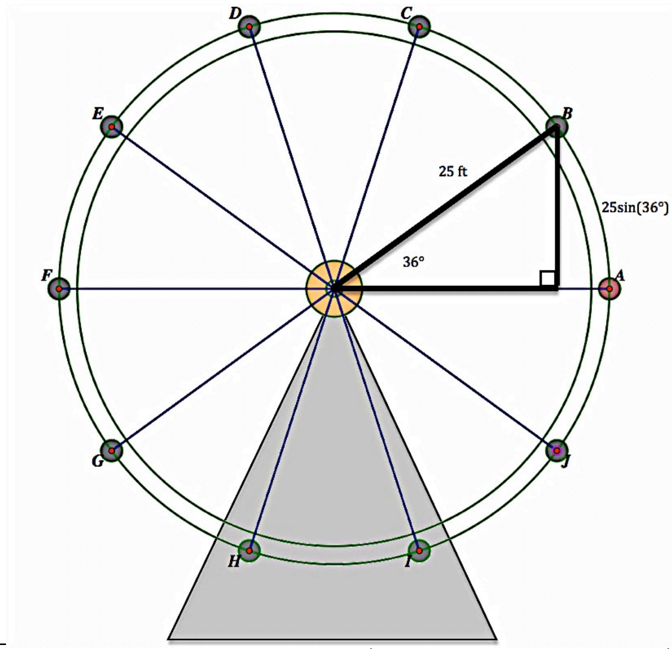
To describe the location of Carlos' shadow as it moves back and forth on the ground beneath him, we could measure the shadow's horizontal distance (in feet) to the right or left of the point directly beneath the center of the Ferris wheel, with locations to the right of the center having positive value and locations to the left of the center having negative values. For instance, in this system Carlos' shadow's location will have a value of 25 when he is at the position farthest to the right on the Ferris wheel, and a value of -25 when he is at a position farthest to the left.

1. What would Carlos' position be on the Ferris wheel when his shadow is located at 0 in this new measurement system?

2. Sketch a graph of the horizontal location of Carlos' shadow as a function of time *t*, where *t* represents the elapsed time after Carlos passes position A, the farthest right position on the Ferris wheel.



3. Calculate the location of Carlos' shadow at the times t given in the following table, where t represents the number of seconds since Carlos passed the position farthest to the right on the Ferris wheel. Keep track of any regularities you notice in the ways you calculate the location of the shadow. As you calculate each location, plot Carlos' position on the diagram at the right.



Elapsed time since passing position A	Calculations	Horizontal position of the rider
1 sec		
2 sec		
3 sec		
4 sec		
8 sec		
14 sec		
18 sec		
23 sec		
28 sec		
40 sec		

4. Write a general formula for finding the location of the shadow at any instant in time.

6.5 Moving Shadows – Teacher Notes

A Practice Understanding Task

Purpose: The purpose of this task is to provide an opportunity for students to practice the ideas, strategies and representations that have surfaced and been examined in the previous tasks, 6.1-6.4. In the context of describing the periodic motion of the shadow of a rider on the Ferris wheel as the shadow moves back and forth across the ground when the sun is directly overhead, students will recognize that the cosine function can be used to measure the distance horizontally from the center of the wheel and they will derive the function *horizontal position of the shadow = 25cos(18t).*

Core Standards Focus:

F.TF.5 Choose trigonometric functions to model periodic phenomena with specified amplitude, frequency, and midline.★

F.TF.2 Explain how the unit circle in the coordinate plane enables the extension of trigonometric functions to all real numbers, interpreted as radian measures of angles traversed counterclockwise around the unit circle. (**Note:** *In this task, the input into the trigonometric function represents an angle of rotation measured in degrees.*)

Related Standards: G.SRT.8Related Standards: F.BF.1c

Standards for Mathematical Practice:

SMP 7 – Look for and make use of structure

The Teaching Cycle:

Launch (Whole Class):

Introduce students to the context of trying to describe the motion of a rider's shadow along the ground when the sun is directly overhead. Make sure students understand the coordinate system we are imposing on this context: a horizontal number line lying on the ground beneath the Ferris wheel with zero located directly beneath the center of the wheel. As usual, distances to the right of zero will be considered positive, and distances to the left of zero will be considered negative. The work on this task is similar enough to previous work that students should need no farther introduction.

Explore (Small Group):

Press students to recognize that the same right triangles they have drawn on the Ferris wheel in previous tasks can be used again to determine the horizontal distances from the center. They should recognize that in this case they want to find the length of the side adjacent to the angle whose vertex is at the center of the wheel, so they will want to use the cosine ratio.

Discuss (Whole Class):

Students will probably assume that a single expression in terms of the cosine can be written to describe the horizontal positions of the shadow of the rider, since we have written a single expression in terms of the sine to represent the height of the rider. Have a student present how they arrived at the function *horizontal position of the shadow = 25cos(18t)* and ask if students are convinced this expression works for all quadrants. Ask students how they are defining *cos(θ)* when $18t$ is not between 0° and 90°.

It is important that students articulate an angle of rotation definition for cosine as follows:

> To define cosine for an angle of rotation θ, select a point (x, y) on the terminal ray and calculate its distance r from the origin using $r = \sqrt{x^2 + y^2}$. The cosine of the angle of rotation is defined by the ratio x/r. In the first quadrant, for angles between 0° and 90°, this ratio is consistent with the right triangle definition of the cosine ratio, since x measures the length of the adjacent side of a right triangle drawn in quadrant I and r measures the length of the hypotenuse. This new definition, however, allows one to define cosine values for angles that measure more than 90° or less than 0°.

Also discuss the following ideas relative to this definition of the cosine. Students should be able to justify why these statements are true.

$\cos(-\theta) = \cos(\theta)$

$\theta + 2\pi n$ or $\theta - 2\pi n$ yield a series of co-terminal angles that have the same cosine value

Aligned Ready, Set, Go: Modeling Periodic Behavior 6.5

READY, SET, GO! Name Period Date

READY

Topic: Comparing radius and arc length

Stage coaches and wagons in the 1800s had wheels that were smaller in the front than in the back. The front wheels had 12 spokes. The top of the front wheel measured 44 inches from the ground. The rear wheels had 16 spokes. The top of the rear wheel measured 59 inches from the ground. (For these problems disregard the hub at the center of the wheel. Assume the spokes meet in the center at a point. **Leave π in your answers.**)

1. Find the area and the circumference of each wheel.

2. Determine the central angle between the spokes on each wheel.

3. Find the distance on the circumference between two consecutive spokes for each wheel.

4. The wagons could cover a distance of 15 miles per day. How many more times would the front wheel turn than the back wheel on an average day?

5. A wheel rotates r times per minute. Write a formula for how many degrees it rotates in t seconds.

SET

Topic: Determining values of cosine in a circle

Use the given point on the circle to find the value of cosine.
Recall $r = \sqrt{x^2 + y^2} and\ cos\ \theta = \frac{x}{r}.$

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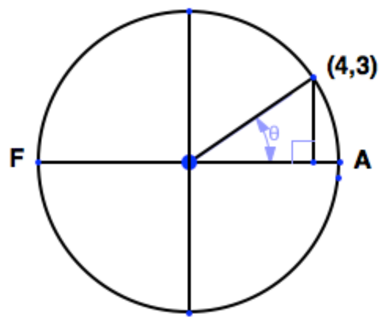

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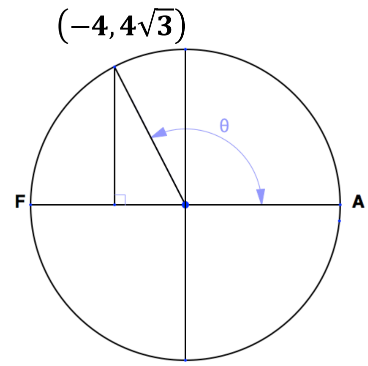

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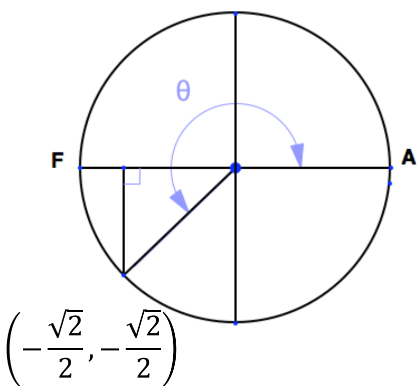

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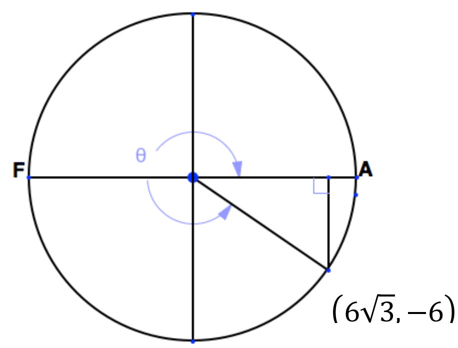
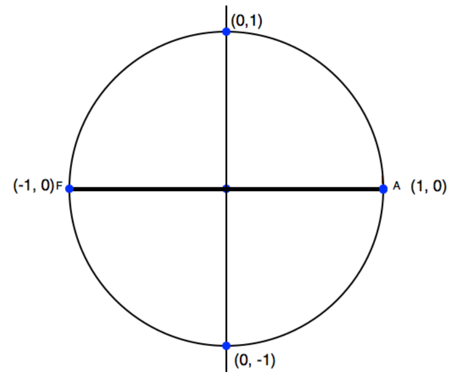


10. In each graph, the angle of rotation is indicated by an arc and θ. Describe the angles of rotation that make the x-values of the points positive and the angles of rotation that make the x-values negative.

11. What do you notice about the x-values and the value of cosine in the graphs?

12. In the graph at the right, the radius of the circle is 1. The intersections of the circle and the axes are labeled. Based on your observation in #11, what do you think the value of cosine might be for the following values of θ:

90°?　　180°?　　270°?　　360°?



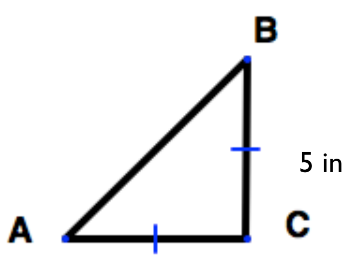
GO

Topic: Reviewing the measurements in special triangles

Use the given information to find the missing sides and the missing angles.

Triangle ABC is a right triangle. Angle C is the right angle. Write the **exact** values for the sides.

13.



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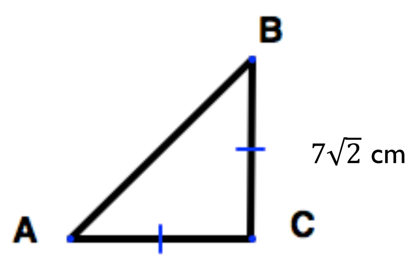

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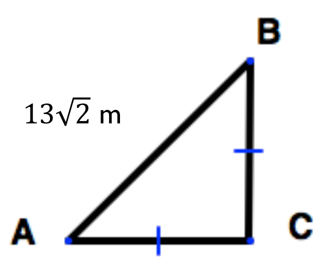

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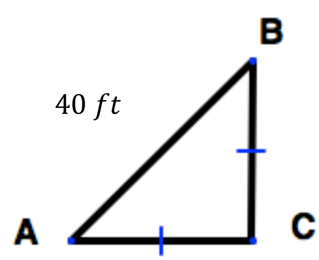

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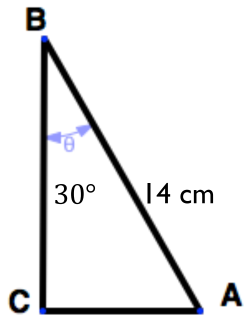

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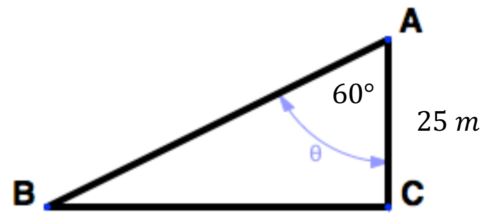

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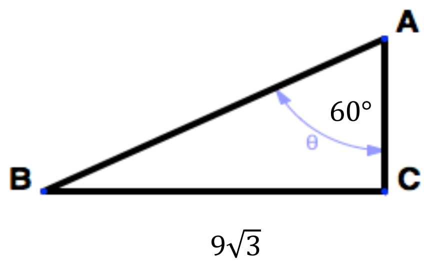

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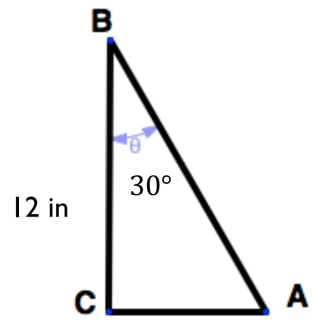

Find AD in the figures below.

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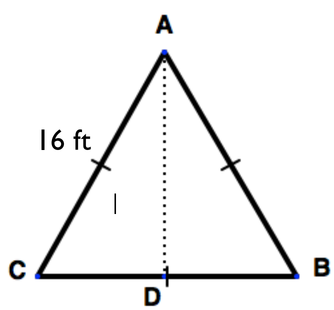

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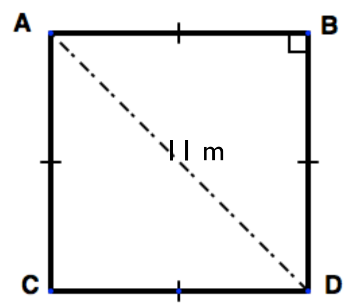

Remember that π is simply a number.

23. If you purchased π gallons of gasoline, about how many gallons of gas did you buy?

24. If you were paid 5π dollars per hour, about how many dollars would you make in 8 hours?

25. If you slept 2π hours each night, about how many hours of sleep would you get per night.

6.6 Diggin' It

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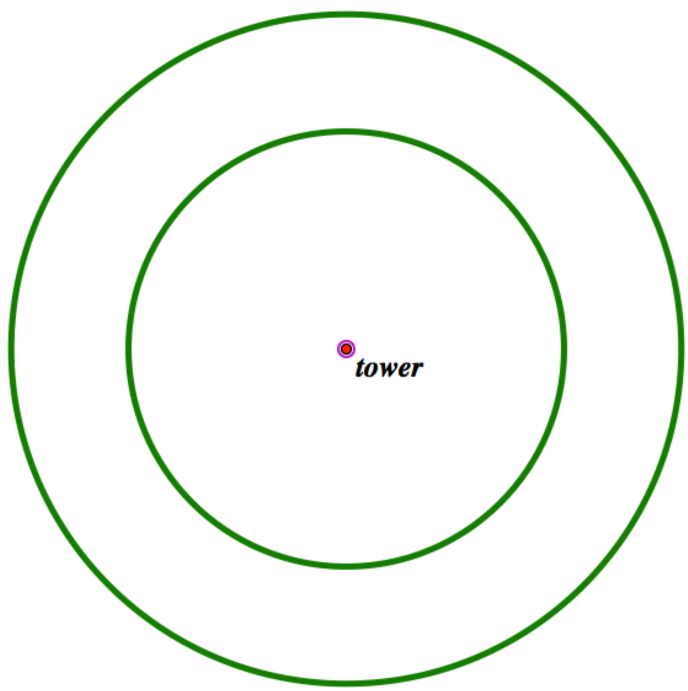
A Develop Understanding Task

Alyce, Javier, and Veronica are responsible for preparing a recently discovered archeological site. The people who used to inhabit this site built their city around a central tower. The first job of the planning team is to mark the site using stakes so they can keep track of where each discovered item was located.

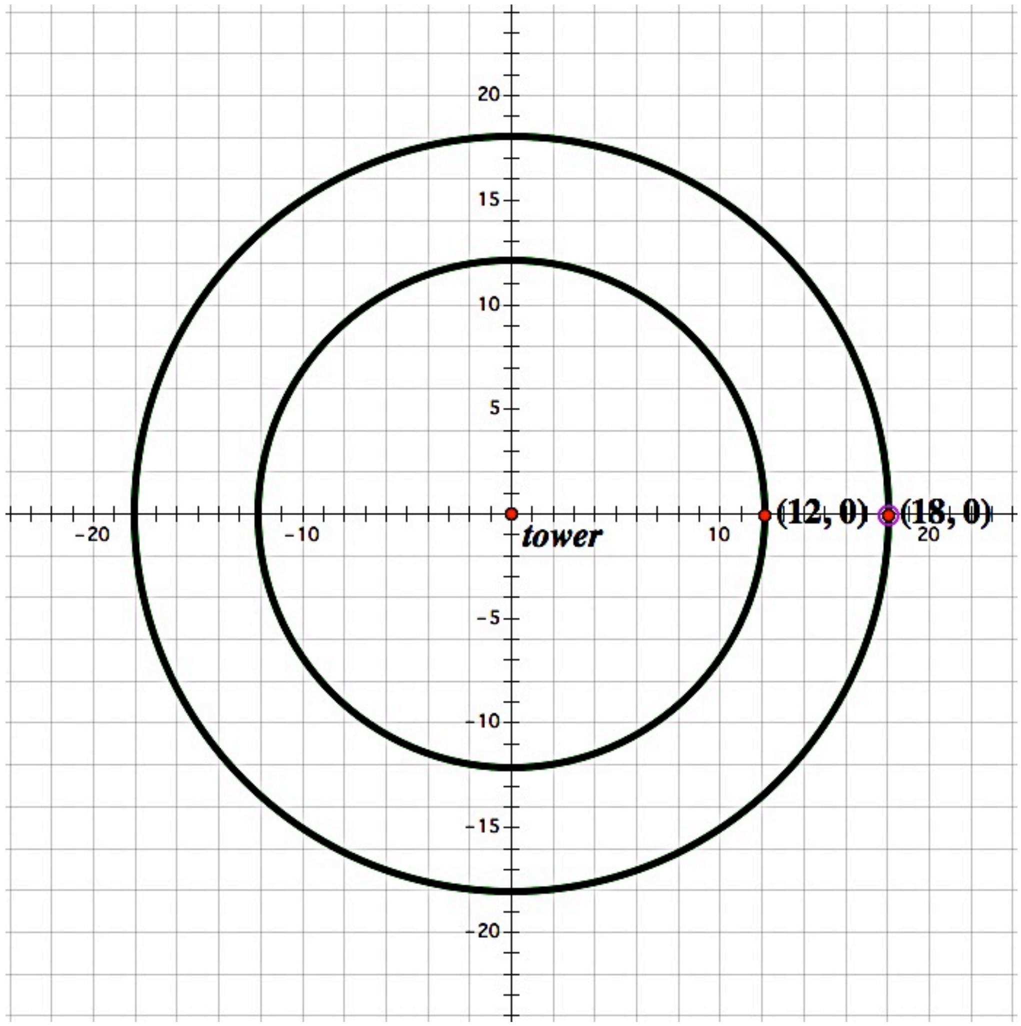
Part 1

1. Alyce suggests that the team place stakes in a circle around the tower, with the distance between the markers on each circle being equal to the radius of the circle. Javier likes this idea but says that using this strategy, the number of markers needed would depend on how far away the circle is from the center tower. Do you agree or disagree with Javier's statement? Explain.

2. Show where the stakes would be located using Alyce's method if one set of markers were to be placed on a circle 12 meters from the center and a second set on a circle 18 meters from the center.



Part 2



3. After looking at the model, Veronica says they need to have more stakes if they intend to be specific with the location of the artifacts. Since most archaeological sites use a grid to mark off sections, Veronica suggests evenly spacing 12 stakes around each circle and using the coordinate grid to label the location of these stakes. The central tower is located at the origin and the first of each set of 12 stakes for the inner and outer circles is placed at the points (12, 0) and (18, 0), respectively. Alyce also wants to make sure they record the distance around the circle to each new stake from these initial stakes. Your job is to determine the x and y-coordinates for each of the remaining stakes on each circle, as well as the arc length from the points (12, 0) or (18, 0), depending on which circle the stake is located. Keep track of the method(s) you use to find these values.

Part 3

Javier suggests they record the location of each stake and its distance around the circle for the set of stakes on each circle. Veronica suggests it might also be interesting to record the ratio of the arc length to the radius for each circle.

4. Help Javier and Veronica complete this table.

	Inner Circle: *r* = 12 meters			**Outer Circle: *r* = 18 meters**		
	Location	**Distance from (12,0) along circular path**	**Ratio of arc length to radius**	**Location**	**Distance from (18,0) along circular path**	**Ratio of arc length to radius**
Stake 1	(12, 0)	0	0	(18, 0)	0	0
Stake 2						
Stake 3						
Stake 4						
Stake 5						
Stake 6						
Stake 7						
Stake 8						
Stake 9						
Stake 10						
Stake 11						
Stake 12						

5. What interesting patterns might Alyce, Javier and Veronica notice in their work and their table? Summarize any interesting things you have noticed.

6.6 Diggin' It – Teacher Notes

A Develop Understanding Task

Purpose: The purpose of this task is to surface alternative ways of measuring a central angle of a circle: in degrees, as a fraction of a complete rotation, or in radians. In this context, students will practice using right triangle trigonometry to find the coordinates of points on a circle. Students will also become more familiar with radian measurement and have a deeper understanding of the relationship between arc length measurements and radian angle measurements.

In Secondary Math II, students encountered the idea that the length of the arc intercepted by an angle is proportional to the radius, and have defined the radian measure of the angle as the constant of proportionality (G.C.5). Part 1 of this task will surface what students understand about these concepts and whether they can connect them to the context of the problem. Part 2 of this task will continue to reinforce students' understanding about right triangle trigonometry from Secondary Math II and show to what extent they are seeing right triangles within the circle, as explored in the previous learning cycle of this module. Part 3 points students' attention towards the radian definition of angle measurement.

Core Standards Focus:

F.TF.1 Understand radian measure of an angle as the length of the arc on the unit circle subtended by the angle.

F.TF.2 Explain how the unit circle in the coordinate plane enables the extension of trigonometric functions to all real numbers, interpreted as radian measures of angles traversed counterclockwise around the unit circle.

Related Standards: G.C.5

Standards for Mathematical Practice:

SMP 7 – Look for and make use of structure

Vocabulary: Students will define *radians* as a way of measuring angles of rotation using the ratio of the *arc length* to the *radius* of the circle on which the angle is being measured.

The Teaching Cycle:

Launch (Whole Class):

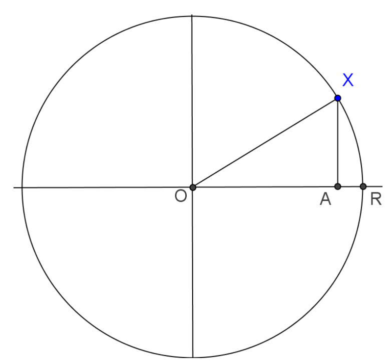
Begin the task by reading the story of the archaeological site. Have students work in pairs to answer questions 1 and 2. At this point, students are using their background knowledge of circumference to think about 'how many radii' fit around a circle. You may want to provide a piece of wire or string that can be cut to the length of the radius of each circle and laid out around the corresponding circle to determine how many segments of the length of the radius fit around the circle, and where each stake would be located using Alyce's strategy. Students should notice that just over 6 radii fit around each circle. Have a student present who has related this experimental result to the formula for the circumference of a circle. If no one has done so, ask students to come up with an explanation as to why this is so, since it may be counterintuitive to many students—those who initially may have agreed with Javier's statement.

Sketch the diagram below on the board and ask the class to consider how many ways we could specify the location of X. Have students discuss possibilities for a minute with a partner. Call on a few pairs to share their ideas with the whole class. Listen for the following ideas to emerge, at least informally:

1) Begin at the origin and go x units from O to A and then y units from A to X, X is at the coordinates (x, y).

2) Begin at the origin and go r units from O to R and go s units along the arc to X, so X is at the coordinates (r, s).

3) Begin at the origin and go r units from O to X at an angle θ from $\overrightarrow{OR}$ so X is at the coordinates (r, θ).



If all of these ideas do not emerge, ask, "Can anyone see another way to specify the location of X?" to elicit the additional ideas. The goal here is simply to activate students' thinking about an important measurement issue—there are many ways to designate the location of X.

Turn students' attention to part 2 of the task. After reading about "your job", call on a few students to describe in their own words what they are aiming to accomplish in this part of the task. Clarify any questions or confusion you may detect.

Explore (Small Group):

As you monitor the work of the groups on Part 2, look for evidence that students understand that:

- The concentric circles are similar, so there will be a relationship between the locations of corresponding stakes on the different circles.
- They will need to use arc length to help them find where to put the stakes.
- The arc length between adjacent stakes will be one-twelfth of the circumference for each circle.

As students come up with strategies, some groups may clearly notice and understand that once they place the stakes carefully on one circle, they can construct a ray from the origin through each stake, extending outward to intersect the other circle and that each point of intersection is a new location for a stake. Other groups may want to measure arcs that are 1/12 of the circumference separately on each circle.

From their work on previous tasks in this module, it is anticipated most students will locate the position of the stakes by considering right triangles, with angles measured in increments of 30°—one-twelfth of a complete rotation around the circle—and using right triangle trigonometry to find the x and y-coordinates of the stakes. Using this strategy they will inevitably have to wrestle with "how far to go East" and "how far to go North" kinds of questions and their right triangle trigonometry understanding should begin to come into play. If groups are settling for visual estimation of distances instead of using trigonometry, press them for more precise strategies.

Students will also have to wrestle with the question, "How far to move along the circle?" and will probably relate this distance to a fraction of the circumference. Watch for students who calculate these distances as decimal numbers versus in terms of multiples of π.

Listen for students who may recall using radians as a way of measuring central angles, perhaps relating this to the work of questions 1 and 2. It is not necessary for radian measurement to emerge at this point in the task, but it would be interesting to note if students recall such work from Secondary Math II (see *Madison's Round Garden*).

Once students have strategies for locating stakes on the circles, both in terms of the x and y coordinates and the arc length s, turn their attention to the table in part 3 of the task. Here they will record their work, and also examine the ratio of arc length to radius—leading to the definition of radian measurement. If time is running short, you may want to assign students to complete this chart as part of their homework. If that is the case, do the first part of the following discussion (sharing strategies for determining x, y and s) before students leave class, and continue the discussion about the table during the following class period.

Discuss (Whole Class):
Select students to present who can explain how they determined the location of the stakes in terms of its x and y-coordinates. One or two examples should suffice. Then have students present strategies for finding the arc length. Include examples of the arc length represented as decimals, and the arc length represented as multiples of π.

Focus the remainder of the whole class discussion on observations students have made during this work, including observations based on the table. Use questions such as the following to get the discussion going.

1. What observations can you make about the locations of the stakes on these circles?
2. Would our observations hold if we changed the number of stakes on each circle?
3. Would our observations change if we changed the size of the circles?
4. What are we noticing in the table?

As students respond, direct the conversation towards developing the idea that the measure of a central angle of a circle can be expressed in many ways: in degrees, as a fraction of a complete rotation (e.g., one-twelfth of a rotation), or in radians—the ratio of arc length to radius ($\theta = s/r$) as shown in the two matching columns in the table and as observed in the work on question 2.
Ask students, "If I use the radius of a circle to measure arc length, like we did in question 2, what fraction of a radius will fit along an arc that is intercepted by a 30° angle?" Use this question to help students start to clarify the concept of radian measurement of angles. It is good for students to recognize that radian measurements can be expressed as decimal numbers, e.g., 0.524 radians, or as multiples of π, e.g., $\pi/6$ radians. Students will have opportunities in the next few tasks to solidify their understanding of radian measurement.

Aligned Ready, Set, Go: Modeling Periodic Behavior 6.6

READY, SET, GO! Name Period Date

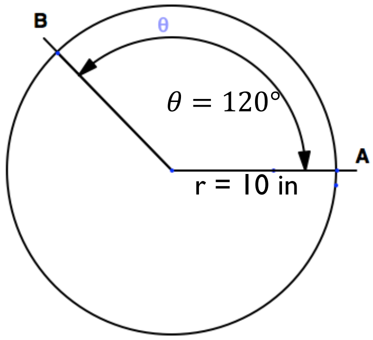
READY

Topic: Finding the length of an arc using proportions

Use the given degree measure of the central angle to set up a proportion to find the length of arc AB.

<u>Leave π in your answers.</u> Recall $s = \frac{\theta}{360°}(d\pi)$ *where s is the arc length.*

1.



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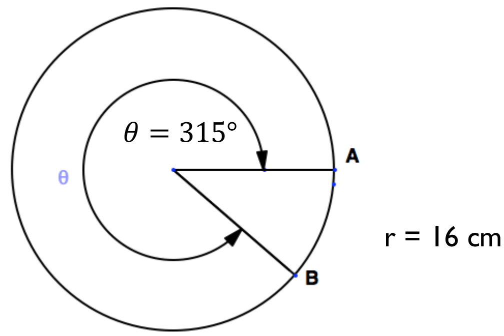

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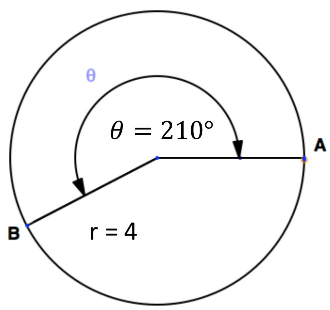

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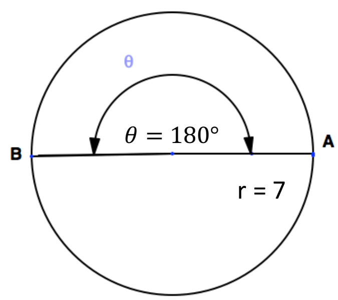

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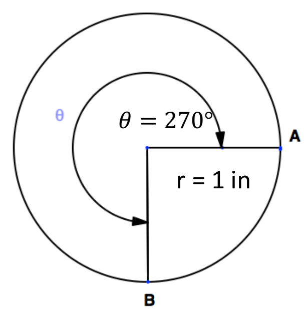


6. The circumference of circle A is 400 meters. The circumference of circle B is 800 meters. What is the relationship between the radius of circle A and the radius of circle B?

   Justify your answer.

SET

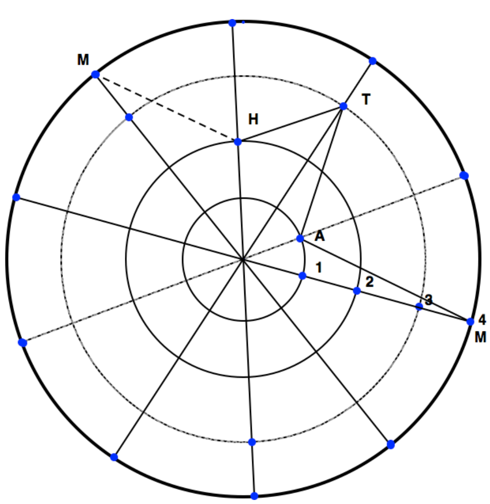
Topic: Describing the location of a point by the angle of rotation and the radius

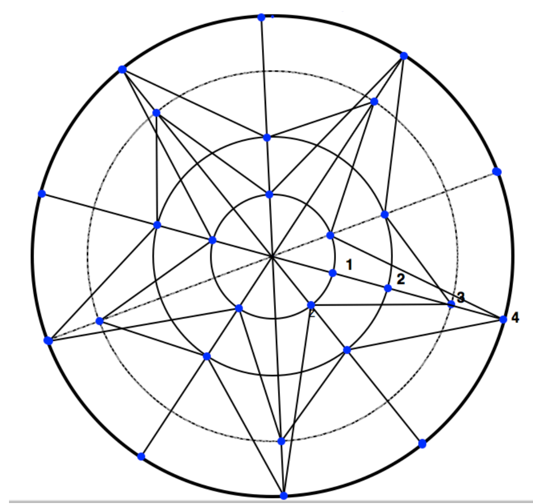
It is possible to identify the location of a point on the edge of a circle in several different ways. One way is to use rectangular coordinates (x, y). In this activity, you will be graphing "words" by using letters to identify points around a circle. The size of the rotation or θ will be the same while the length of the radius will change. First select a word. Avoid words containing 5 letters or multiples of 5. I am choosing the word MATH. Assign a number to each letter of your word according to the table below. The numbers correspond to the concentric circles. You can begin on any spoke. Move from one spoke to the next in a positive rotation. Make a dot at the intersection of the spoke and the circle corresponding with the number of the letter you are on. You will need to make more than one rotation of the circle in order to close your figure.

Circle numbers and their corresponding letters. The letters for "MATH" are high-lighted.

Circle 1: A,D,K,L,N,V,Z	**Circle 2: E, U, G, H, Q, U, X**
Circle 3: I, C, F, J, T, S, Y	**Circle 4: O, B, E, M, R, P, W**

The word MATH will use the numbered circles 4 1 3 2 in that order. You can begin on any spoke. I began on the spoke with the numbers. I made a dot on 4, rotated to the next spoke and made a dot on 1. I connected the two dots. Then I moved to circle 3, made a dot, connected the segment, and moved to circle 2. You can see MATH marked on the diagram. After marking H, I started over with M on the next spoke. (See the dotted line.) Continue spelling MATH and rotating around the circle until the figure is closed and the path repeats itself. The figure at the right is the completed graph of the word MATH. I always knew MATH was beautiful!

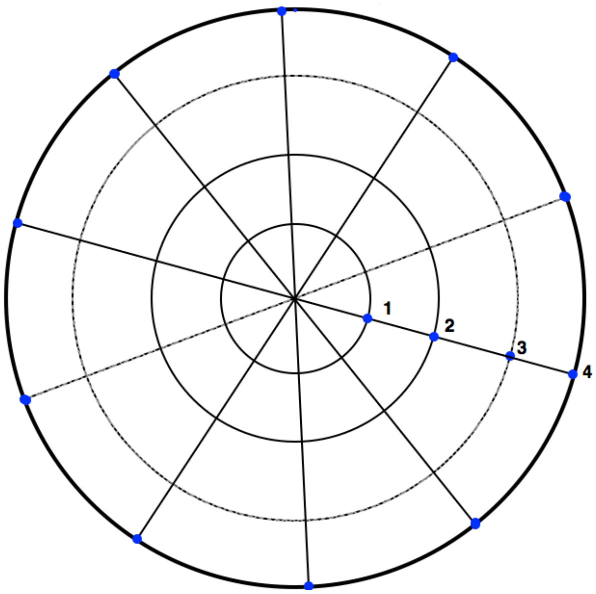




Now it's your turn. Select a word. Short ones are best. Assign the numbers and begin.

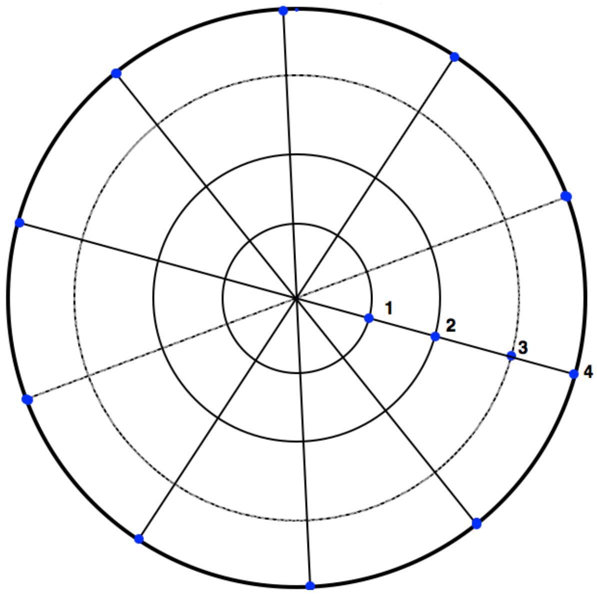
7.

word ________________________________



8.

word ________________________________



9. What is the angle between each spoke in the grid above?

10. How many degrees did it take to graph MATH once? (From M to H?)

11. How many degrees did it take to graph MATHM? (From M to the M again)

12. How many times did I need to spell the word MATH to complete the graph?

13. How many rotations did it take?

Can you figure out the answer to this question without counting? Explain.

GO

Topic: Converting angles between radians and degrees

Recall that there are 360 **degrees** in a full circle and 2π **radians** in a full circle. Therefore, $360º = 2\pi$ radians. If we divide both sides of the equation by 2, we create another identity $180º = 1\pi\ radians$. We can use this identity to convert degrees to radians or radians to degrees.

Since $180° = \pi\ radians$, it follows that $\frac{\pi\ radians}{180°} = \frac{180°}{\pi\ radians} = 1$.

If I want to convert 72º into radian measure, then I need the unit of degrees to cancel, so I will multiply 72º by $\frac{\pi\ radians}{180°}$, example: $72° \cdot \frac{\pi\ radians}{180°} = \frac{72º \times \pi\ radians}{180º} = \frac{2\pi\ radians}{5}$.

The unit *radians* is usually left off. Hence, an angle that measures 72º is equivalent to a radian measure of $\frac{2\pi}{5}$.

Convert the following angles from degrees to radians or radians to degrees.

14. 45°

15. 15°

16. 54°

17. 135°

18. 300°

19. 270°

20. $\frac{5\pi}{6}$

21. $\frac{\pi}{8}$

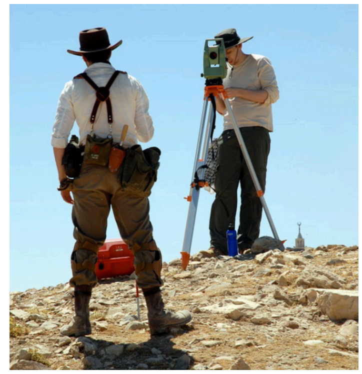
22. $\frac{3\pi}{4}$

23. $\frac{7\pi}{5}$

24. $\frac{\pi}{18}$

25. $\frac{13\pi}{12}$

6.7 Staking It



A Solidify Understanding Task

After considering different plans for laying out the archeological site described in *Diggin' It,* Alyce, Javier and Veronica have decided to make concentric circles at 10-meter intervals from the central tower. They have also decided to use 16 stakes per circle, in order to have a few more points of reference. Using ropes of different lengths to keep the radius constant, they have traced out these circles in the sand. Because they know the circles will soon be worn away by the wind and by people's footprints, they feel a sense of urgency to locate the positions of the 16 stakes that will mark each circle. The team wants to be efficient and make as few measurements as possible.

Part 1

Veronica suggests they should locate the stakes around one circle and use those positions to mark where the stakes will go on all of the other circles.

1. What do you think about Veronica's idea? How will marking stake positions on one circle help them locate the positions of the stakes on all of the other circles?

Veronica has decided they should stake out the circle with a radius of 50 meters first. She is standing at the point (50,0) and knows she needs to move 22 ½° around the circle to place her next stake. But, she wonders, "How far is that?"

- Veronica decides she will find the distance by setting up a proportion using degree measurements.
- Alyce thinks they should find the distance by taking $^{1}/_{16}$ of the circumference.
- Javier thinks they should use radian measurement in their calculation.

2. Show how each team member will calculate this distance.

   Veronica's Strategy

   Alyce's Strategy

   Javier's Strategy

Part 2

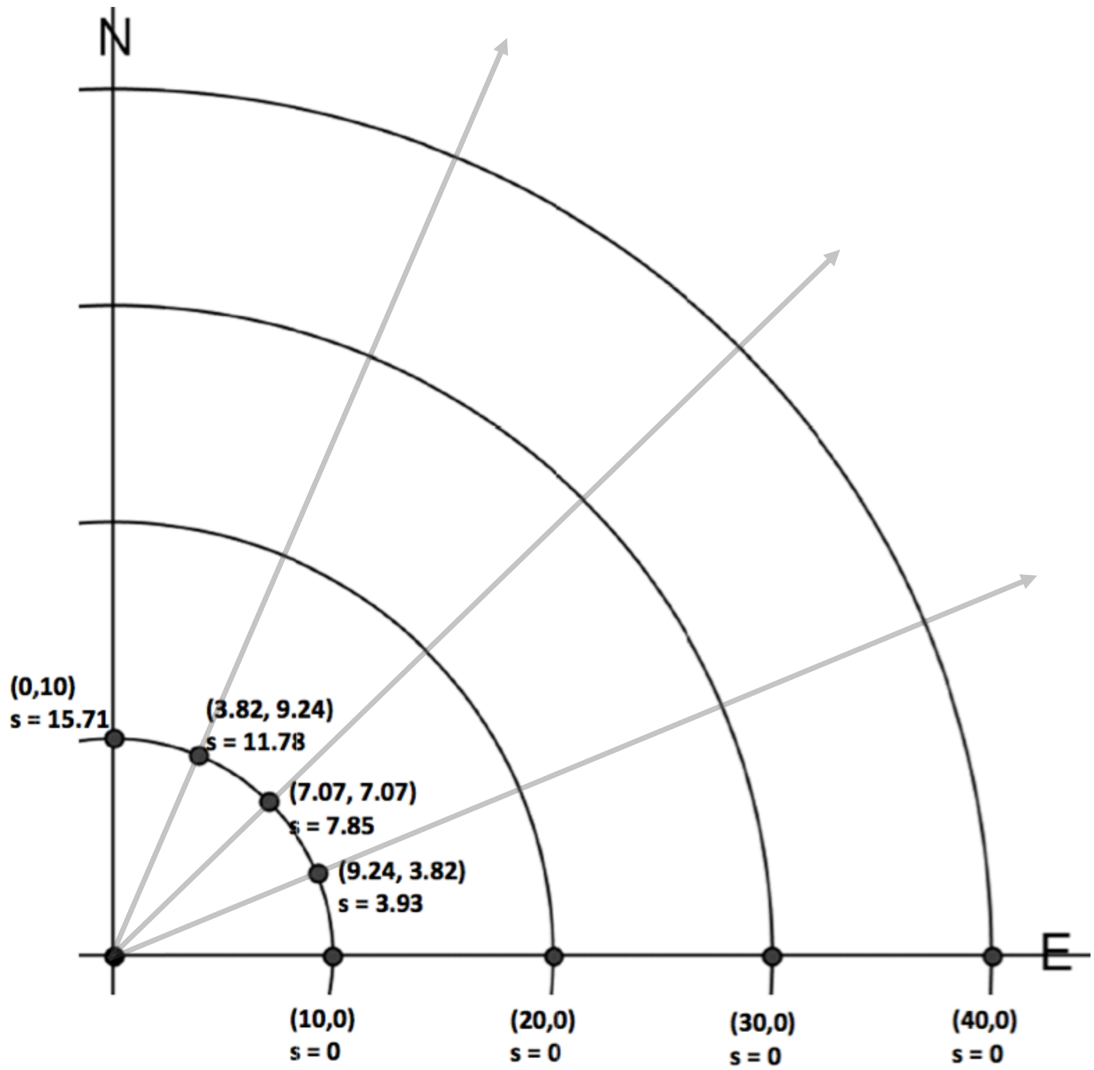
Javier has a different idea. He suggests they should figure out the locations of all of the stakes in quadrant I first, and then it would be easy to find the locations of the stakes in all the other quadrants by using the quadrant I locations.

3. What do you think about Javier's suggestion? How will marking the location of stakes in quadrant I help them figure out the location of the stakes in other quadrants?

Javier has already started working on his strategy and has completed the calculations for the 10-meter circle. (see Javier's diagram).

4. Develop a strategy to locate all of the other stakes in the first quadrant for these additional circles. Find the coordinates and arc lengths for each. Describe the strategy you used to make the fewest calculations for finding the coordinates and arc lengths for these additional stakes.

Javier's Diagram



6.7 Staking It – Teacher Notes

A Solidify Understanding Task

Purpose: The purpose of this task is to solidify students' previous understanding of radians—that is, the radian measure of an angle is the ratio of the length of an intercepted arc to the radius of the circle on which that arc lies, and to use radian measurement as a proportionality constant in computations. This task and the previous task lay a foundation for a new understanding of radians—that is, the radian measure of an angle is the length of the intercepted arc on a unit circle. This latter idea will be solidified in the next task of this learning cycle.

Core Standards Focus:
F.TF.1 Understand radian measure of an angle as the length of the arc on the unit circle subtended by the angle.

F.TF.2 Explain how the unit circle in the coordinate plane enables the extension of trigonometric functions to all real numbers, interpreted as radian measures of angles traversed counterclockwise around the unit circle.

Related Standards: G.C.5

Standards for Mathematical Practice:
SMP 7 – Look for and make use of structure

Vocabulary: Students will continue examining the definition of *radians* as a way of measuring angles of rotation using the ratio of the *arc length* to the *radius* of the circle on which the angle is being measured.

The Teaching Cycle:

Launch (Whole Class): Part 1

After reading through the scenario for this task have students discuss question 1 about Veronica's strategy for placing the stakes on concentric circles. This question is intended to help students focus on the relationship between arc length and angle measure as we move from one concentric circle to another. Students should recognize that regardless of the circle, 16 stakes equally spaced around each circle would form a set of 16 radial lines—like spokes on a Ferris wheel. The angle measure between each of these radial lines is the same, even though the lengths of the intercepted arcs on each circle are different. This is an essential understanding for using radian measurement. After students have discussed this idea, have them work on question 2 in groups of three, with each person in the group taking the lead role in representing either Veronica's, Alyce's or Javier's strategy. Ask students to calculate the distance to the next stake accurate to the nearest tenth of a meter. This will allow them to compare decimal answers obtained using each strategy.

Explore (Small Group): Part 1

While individual students in each group of three will calculate the distance to the next stake using a different strategy, it is important that the students in each group understand each strategy. Make sure each group has a conversation comparing and contrasting each of the three strategies. If students get different results using different strategies have them resolve the differences by examining each other's work. Select students from different groups to present Veronica's, Alyce's and Javier's strategies.

Discuss (Whole Class): Part 1

Have students present each strategy:

Veronica's strategy: $\frac{22.5°}{360°} = \frac{d}{100\pi} \quad \Longrightarrow \quad d = 19.6\ meters$

Alyce's strategy: $d = \frac{1}{16}(100\pi) = 19.6\ meters$

Javier's strategy: $S = \mathrm{r} \cdot \theta \quad \Longrightarrow \quad 50 \cdot \frac{2\pi}{16} = 19.6\ meters$

Ask students to explain why all of these strategies are equivalent, perhaps by looking at how 2π and $^1/_{16}$ are embedded in each calculation.

Launch (Whole Class): Part 2

Discuss question 3 as a whole class. From the work in the previous learning cycle students should be able to address how they can use the symmetry of the circle and related reference triangles to explain why finding coordinates of the stakes in the first quadrant would allow them to find the coordinates of the location of stakes in the other three quadrants. Reinforce this thinking by having them give the coordinates of the stakes in quadrants II, III and IV that correspond to the three given stakes in quadrant I. Assign students to work in their groups of three on question 4 where they will determine the coordinates of the stakes in quadrant I that lie on the circles of radius 20, 30 and 40 meters. Each student in the group should find the coordinates of the stakes on a different radial line.

Explore (Small Group): Part 2

Watch for strategies students use to calculate the coordinates of each stake. If they are using Veronica's, Alyce's or Javier's strategies from part 1, ask if there is another way to do this. Listen for students who recognize (or come to recognize after doing a calculation or two) that they can just double or triple the coordinates or arc length given on the 10-meter circle to get the coordinates or arc length on the 20-meter or 30-meter circles.

Discuss (Whole Class): Part 2

As a whole class, examine the detailed calculations for locating a stake on one of the other circles and compare this to the calculations that would have been used to calculate the given values for the corresponding stake on the circle of radius 10. Ask students to use these computations to explain why the coordinates or arc length of stakes along a radial line are just constant multiples of each other.

Aligned Ready, Set, Go: Modeling Periodic Behavior 6.7

READY, SET, GO! Name Period Date

READY

Topic: Finding the coordinates of points on a circle

Given the equation of a circle centered at (0,0), find one point in each quadrant that lies on the given circle.

1. $x^2 + y^2 = 25$ quadrant I ______________ quadrant II ______________
   quadrant III ______________ quadrant IV ______________

2. $x^2 + y^2 = 4$ quadrant I ______________ quadrant II ______________
   quadrant III ______________ quadrant IV ______________

3. $x^2 + y^2 = 36$ quadrant I ______________ quadrant II ______________
   quadrant III ______________ quadrant IV ______________

4. $x^2 + y^2 = 1$ quadrant I ______________ quadrant II ______________
   quadrant III ______________ quadrant IV ______________

5. $x^2 + y^2 = 9$ quadrant I ______________ quadrant II ______________
   quadrant III ______________ quadrant IV ______________

SET

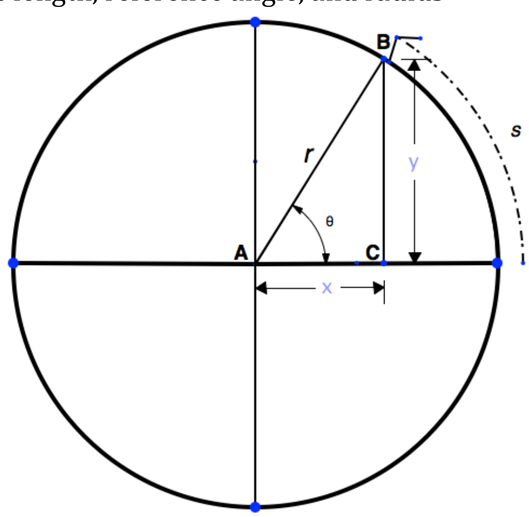
Topic: Locating points in terms of rectangular coordinates, arc length, reference angle, and radius

In the diagram triangle ABC is a right triangle.

Point B lies on the circle and is described by the rectangular coordinates (x, y).

s is the length of the arc subtended by angle θ.

r is the radius of circle A.



Use the given information to answer the following questions.

6. B has the rectangular coordinates (5, 12).

a. Find r.

b. Find θ to the nearest tenth of a degree.

c. Find s by using the formula $s = \frac{\theta}{360°}(d\pi)$.

d. Describe point B using the coordinates (r, θ).

e. Describe point B using the radius and arc length (r, s).

7. B has the rectangular coordinates (33, 56).

a. Find r.

b. Find θ to the nearest tenth of a degree.

c. Find s by using the formula $s = \frac{\theta}{360°}(d\pi)$.

d. Describe point B using the coordinates (r, θ).

e. Describe point B using the radius and arc length (r, s).

8. B is described by (r, θ)

$$where\ \ \theta \approx 58.11°\ and\ r = 53.$$

a. Find (x, y) to the nearest whole number.

b. Find s by using the formula $s = \frac{\theta}{360°}(d\pi)$.

c. Describe point B using the radius and arc length (r, s).

9. B is described by (r, θ)

$$where\ \ \theta \approx 25.01°\ and\ r = 85.$$

a. Find (x, y) to the nearest whole number.

b. Find s by using the formula $s = \frac{\theta}{360°}(d\pi)$.

c. Describe point B using the radius and arc length (r, s).

10. B is described by (r, s)

$$where\ \ s \approx 46\ and\ r = 37.$$

a. Find (x, y) to the nearest whole number.

b. Find θ by using the formula $s = \frac{\theta}{360°}(d\pi)$.

c. Describe point B using (r, θ)

11. B is described by (r, s)

$$where\ \ s \approx 62.26\ and\ r = 73.$$

a. Find (x, y) to the nearest whole number.

b. Find θ by using the formula $s = \frac{\theta}{360°}(2\pi r)$.

c. Describe point B using (r, θ)

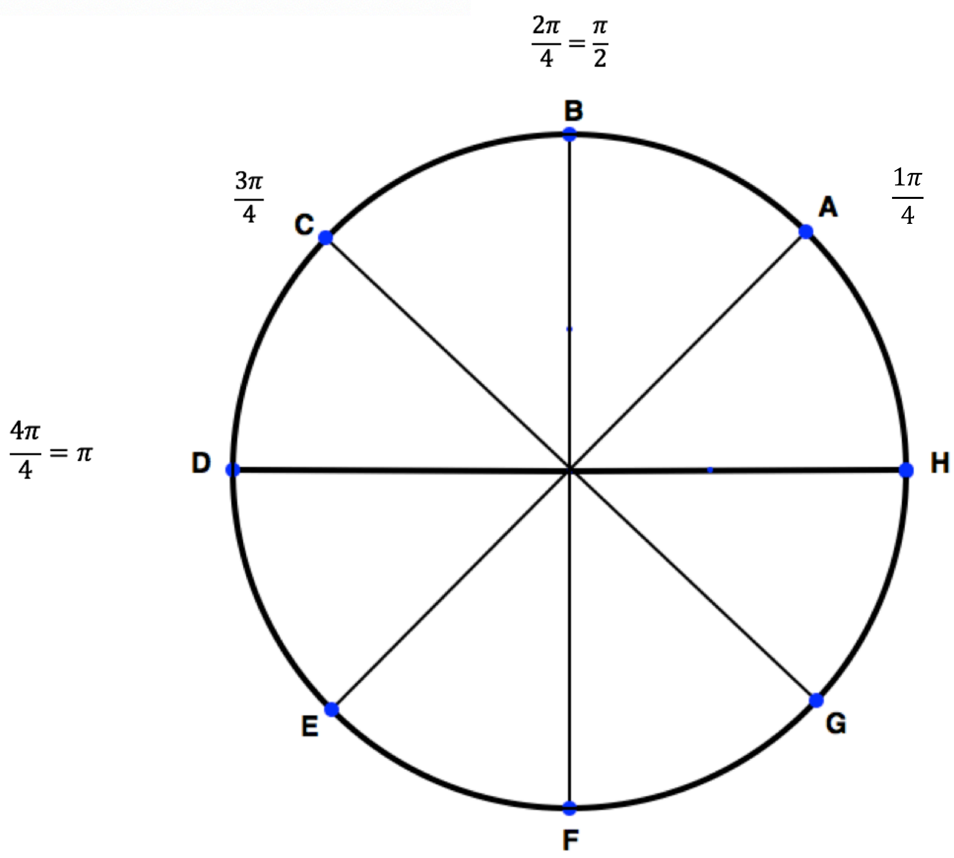
GO

Topic: Making sense of radian measure

Label each point on the circle with the measure of the angle of rotation. Angle measures should be in radians. (Recall a full rotation around the circle would be 2π radians.)

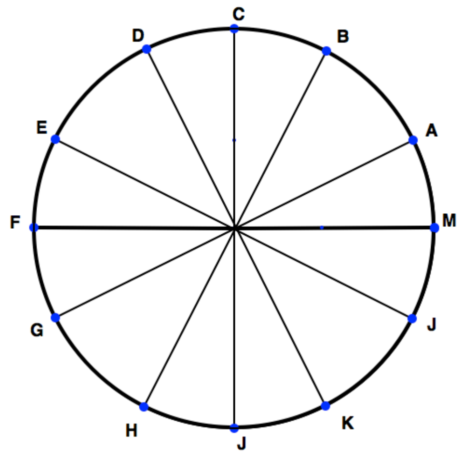
Example: The circle has been divided equally into 8 parts. Each part is equal to $\frac{2\pi}{8}$ *or* $\frac{\pi}{4}$ radians. Indicate how many parts of $\frac{\pi}{4}$ radians there are at each position around the circle.

Finish the example by writing the angle measures for points E, F, G, and H.



Label the figures below using a similar approach as the example.

12.



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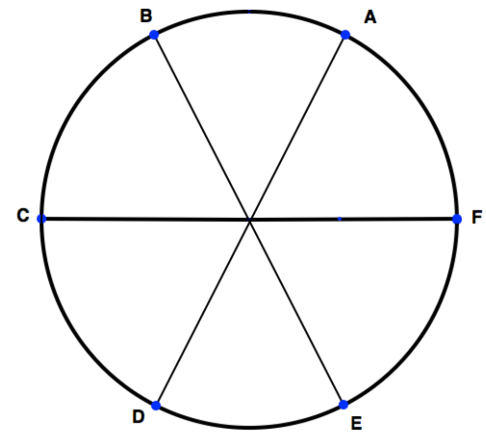

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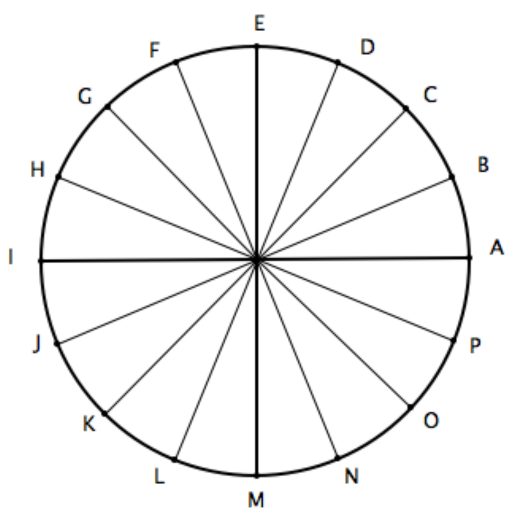

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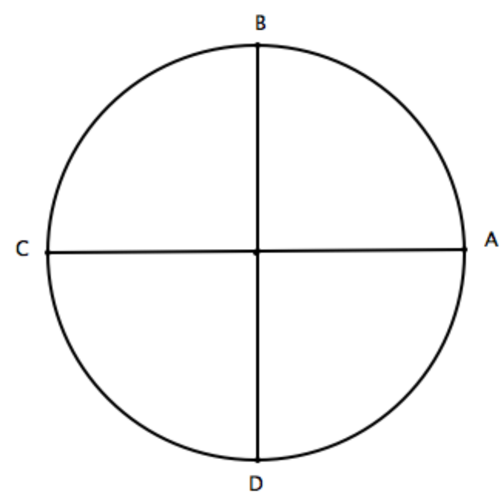

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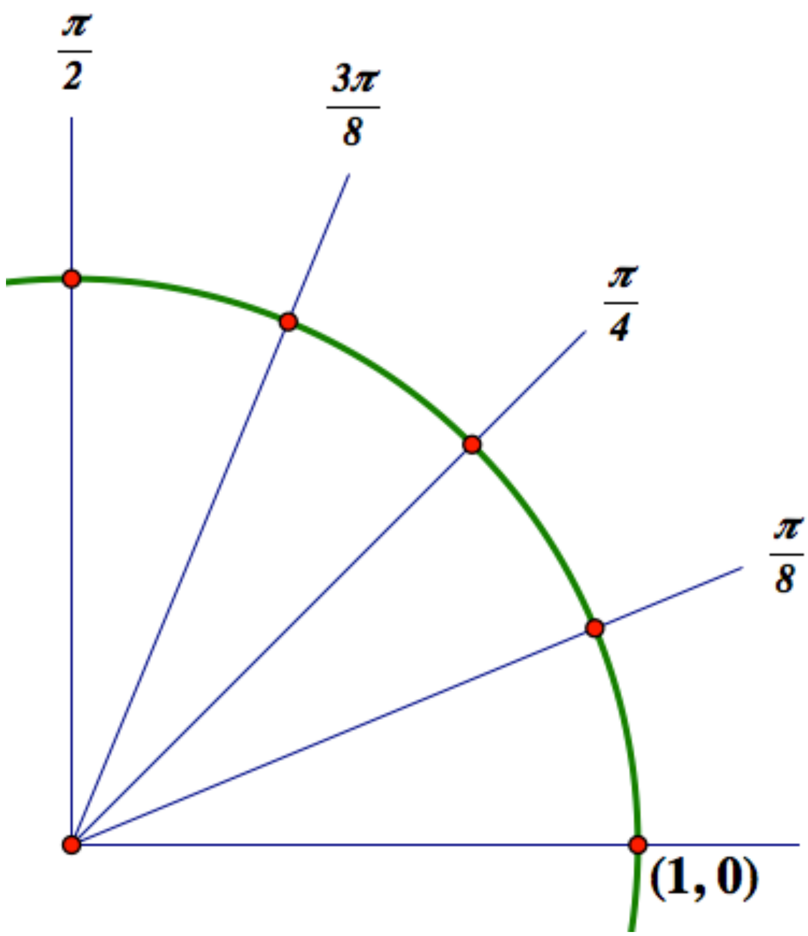
6.8 "Sine"ing and "Cosine"ing It

A Solidify Understanding Task

In the previous tasks of this module you have used the similarity of circles, the symmetry of circles, right triangle trigonometry and proportional reasoning to locate stakes on concentric circles. In this task we consider points on the simplest circle of all, the circle with a radius of 1, which is often referred to as "*the unit circle*."

Here is a portion of a unit circle—the portion lying in the first quadrant of a coordinate grid. As in the previous task, *Staking It*, this portion of the unit circle has been divided into intervals measuring $\frac{\pi}{8}$ radians. As in the previous task, find the coordinates of each of the indicated points in the diagram. Also find the arc length, *s*, from the point (1, 0) to each of the indicated points.



Javier has been wondering if his calculator will allow him to calculate trigonometric values for angles measured in radians, rather than degrees. He feels like this will simplify much of his computational work when trying to locate the coordinates of stakes on the circles surrounding the central tower of the archeological site.

After consulting his calculator's manual, Javier has learned that he can set his calculator in radian mode. After doing so, he is examining the following calculations.

1. With your calculator set in radian mode, find each of the following values. Record your answers as decimal approximations to the nearest thousandth.

$\sin\left(\frac{\pi}{8}\right) =$ $\cos\left(\frac{\pi}{8}\right) =$ $\frac{\pi}{8} =$

$\sin\left(\frac{\pi}{4}\right) =$ $\cos\left(\frac{\pi}{4}\right) =$ $\frac{\pi}{4} =$

$\sin\left(\frac{3\pi}{8}\right) =$ $\cos\left(\frac{3\pi}{8}\right) =$ $\frac{3\pi}{8} =$

$\sin\left(\frac{\pi}{2}\right) =$ $\cos\left(\frac{\pi}{2}\right) =$ $\frac{\pi}{2} =$

2. The coordinates and arc lengths you found for points on the unit circle seem to be showing up in Javier's computations. Why is that so? That is, . . .

   - explain why the radian measure of an angle on the unit circle is the same as the arc length?

   - explain why the sine of an angle measured in radians is the same as the *y*-coordinate of a point on the unit circle?

- explain why the cosine of an angle measured in radians is the same as the *x*-coordinate of a point on the unit circle?

3. Based on this work, find the following without using a calculator:

$\sin\left(\frac{5\pi}{8}\right) =$ $\cos\left(\frac{7\pi}{8}\right) =$ $\cos(\pi) =$

6.8 "Sine"ing and "Cosine"ing It – Teacher Notes

A Solidify Understanding Task

Purpose: While previous tasks have solidified students' understanding of radians as the ratio of the length of an intercepted arc to the radius of the circle on which that arc lies, this task builds a new understanding of radians —the radian measure of an angle is the length of the intercepted arc on a unit circle. Students will also use radians, rather than degrees, to find trigonometric values for angles measured in radians, and will observe that the x and y-coordinates of points on the unit circle correspond with the sine and cosine of the angle of rotation measured from the ray passing through the point (1, 0).

Core Standards Focus:

F.TF.1 Understand radian measure of an angle as the length of the arc on the unit circle subtended by the angle.

F.TF.2 Explain how the unit circle in the coordinate plane enables the extension of trigonometric functions to all real numbers, interpreted as radian measures of angles traversed counterclockwise around the unit circle.

Related Standards: G.C.5

Standards for Mathematical Practice:

SMP 7 – Look for and make use of structure

Vocabulary: In this task students will work with angles of rotation drawn in standard position on the *unit circle*, the circle centered at the origin with a radius of 1 unit. The equation of the unit circle is $x^2 + y^2 = 1$.

The Teaching Cycle:

Launch (Whole Class):

Inform students that they may use their work from the previous task as they work on finding the coordinates and arc length for the points in the first quadrant of the unit circle diagram.

Explore (Small Group):

Listen for student strategies. It is anticipated that students will recognize the radius of this circle is $^1/_{10}$ of the radius of the smallest circle on the previous task, and therefore fill out the coordinates and arc lengths using that fact. Other students may use the definitions of sine and cosine to calculate the coordinates, or one of the student strategies from the previous task to calculate the arc lengths. Regardless of the strategies used, they should notice that the calculator produces values for the cosine and sine of the given angles that are the same as the calculated values of the x and y-coordinates of the points on the unit circle, and that the arc lengths on the unit circle are the same as the measure of the angles in radians. This should be a surprising result at first, until the students consider the definitions of sine, cosine and radian measure more carefully. Listen for students who make and can explain this observation, and students who can apply this observation to use the points on the unit circle as a "trig table" for angles that are multiples of the given angles: $^{\pi}/_{8}$, $^{\pi}/_{4}$, $^{3\pi}/_{8}$.

Discuss (Whole Class):

First, discuss the observation that the x and y-coordinates of points on the unit circle <u>are</u> the trig values of the angles of rotation that get us to those points, and that the angle of rotation measured in radians <u>is</u> the arc length. Plot a generic point on the unit circle diagram and label the coordinates of the point with the ordered-pair $(\cos(\theta), \sin(\theta))$ and mark the angle of rotation θ. Make sure students can explain why points on the unit circle can be labeled this way. For example, they might say, "Since r = 1 on the unit circle, the ratio $\frac{y}{r}$ reduces to y, so $\sin(\theta) = y$." Or they might say, "Since r = 1 on the unit circle, the arc length $S = r\theta$ becomes $S = \theta$.

Second, discuss how the unit circle might be used as a tool for finding trig values of angles of rotation that are multiples of those angles given in the first quadrant. Have students share how they found the three trig values asked for at the end of the task. Make sure their explanations are related to the unit circle.

Aligned Ready, Set, Go: Modeling Periodic Behavior 6.8

READY, SET, GO! Name Period Date

READY

Topic: Reducing complex fractions

Write each of the following as a simple fraction. Rationalize the denominators when appropriate.

1. $\dfrac{\frac{1}{\sqrt{2}}}{\frac{1}{\sqrt{2}}}$

2. $\dfrac{\frac{8\sqrt{3}}{5}}{\frac{1}{5}}$

3. $\dfrac{8}{\frac{1}{2}}$

4. $\dfrac{\frac{7\sqrt{3}}{2}}{\frac{1}{2}}$

5. $\dfrac{1}{\sqrt{2}}$

6. $\dfrac{3}{\sqrt{3}}$

7. $\dfrac{4}{\sqrt{8}}$

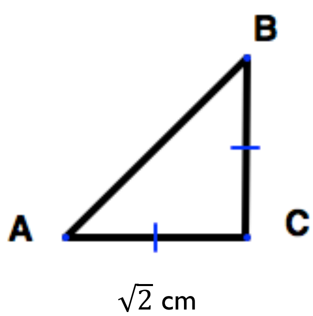
8. $\dfrac{\frac{2}{3}}{\frac{1}{2}}$

9. $\dfrac{\frac{2}{\sqrt{7}}}{\frac{5}{\sqrt{7}}}$

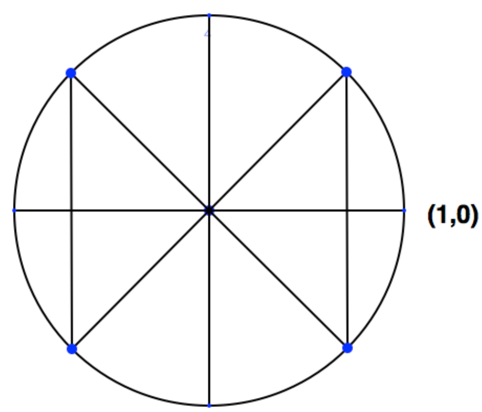
SET

Topic: Finding sine of an angle in radian measure

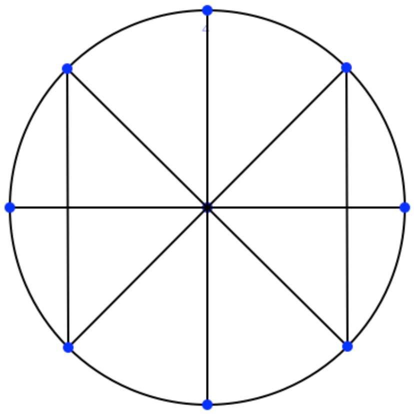
10. Triangle ABC is an isosceles right triangle. The length of one side is given. Fill in the values for the missing sides and angles A and B.



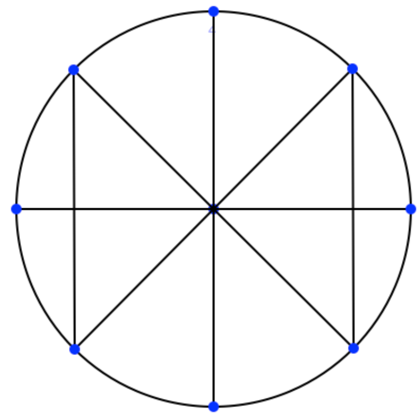
11. Label each point around the circle with the angle of rotation in radians starting from the point (1,0). (each section is equal)



12. Use the values in #10 to write the **exact** coordinates of the 4 points on the circle below. Be mindful of the numbers that are negative.



13. Find the arc length, *s,* from the point (1,0) to each point around the circle. Record your answers as decimal approximations to the nearest thousandth.



Use your calculator to find the following values.

14. $\sin\frac{5\pi}{4} =$

15. $\sin\frac{7\pi}{4} =$

16. Why are both of your answers negative?

17. $\cos\frac{\pi}{4} =$

18. $\cos\frac{7\pi}{4} =$

19. Why are both of your answers positive?

20. $\sin\frac{3\pi}{4} =$

21. $\cos\frac{3\pi}{4} =$

22. Why is one answer positive and one answer negative?

GO

Topic: Recalling trigonometric values of special triangles

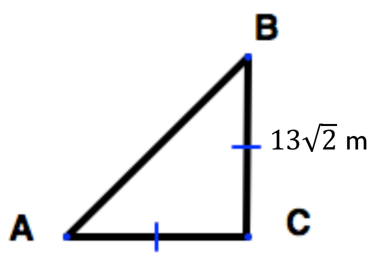
Angle C is the right angle in each of the triangles below. **Use the given information to find the missing sides and the missing angles. Then find the indicated trig values.** Rationalize denominators when appropriate. Do NOT change the values to decimals. Square roots are **exact** values. Decimal representations of the square roots are approximations.

23.

sin A =

cos A =

tan A =

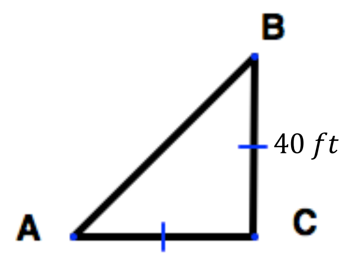


24.

sin B =

cos B =

tan B =



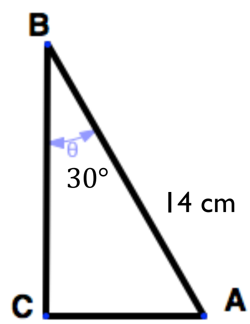
25. Explain why the trig values were the same for angle A and angle B even though the dimensions of the triangles were different.

26.

sin B =

cos B =

tan B =

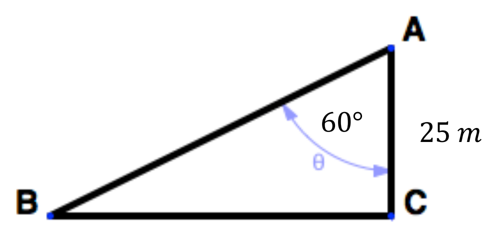


27.

sin A =

cos A =

tan A =

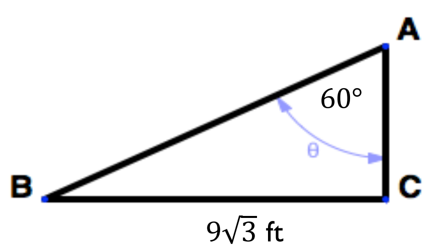


28.

sin A =

cos A =

tan A =

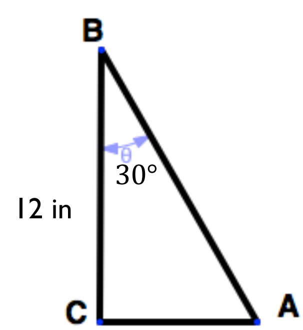


29.

sin B =

cos B =

tan B =



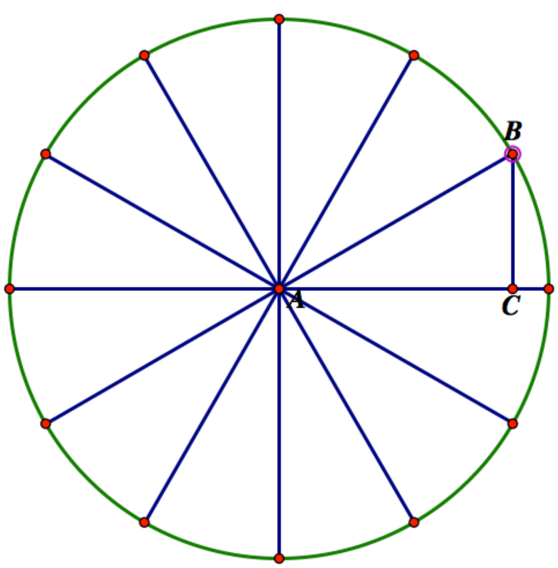
30. Explain where you see the meaning of the identity $\sin\theta = \cos(90° - \theta)$ in problems 26, 27, 28, and 29.

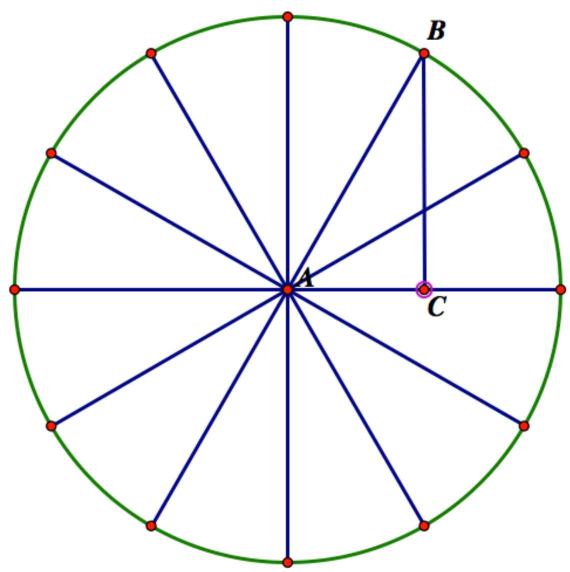
6.9 Water Wheels and the Unit Circle

A Practice Understanding Task

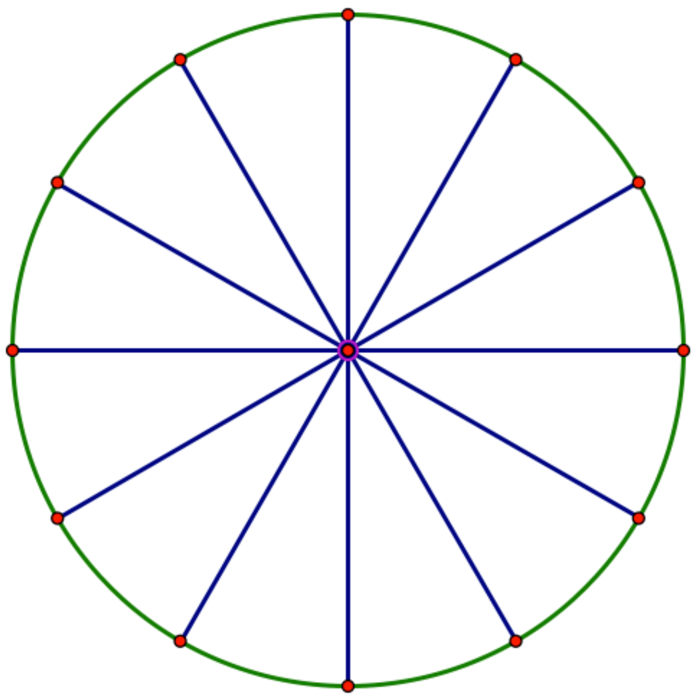
Water wheels were used to power flour mills before electricity was available to run the machinery. The water wheel turned as a stream of water pushed against the paddles of the wheel. Consequently, unlike Ferris wheels that have their centers above the ground, the center of the water wheel might be placed at ground level, so the lower half of the wheel would be immersed in the stream.

1. The following diagrams show potential designs for a water wheel. Each of the 12 spokes of the water wheel will measure 1 meter. In addition to the spokes, the designer wants to add braces to provide additional strength. Two potential placements for the braces are shown in the following diagrams. (The braces and the spoke to which they are attached form a right angle.)
   - Find the measures of $\angle BAC$ and $\angle ABC$ in each diagram.
   - Find the <u>exact lengths</u> of $\overline{AB}$, $\overline{AC}$ and $\overline{BC}$, not just decimal approximations. Explain how you found these lengths exactly.
   - Label the exact coordinates of point B in each diagram.





2. Based on your work above, label the exact values of the *x* and *y*-coordinates for each point on the following schematic drawing of the water wheel. Remember that the center of the wheel is at ground level, so points below the center of the wheel should be labeled with negative values. As in the Ferris wheel models, label points to the left of center with negative coordinates also.



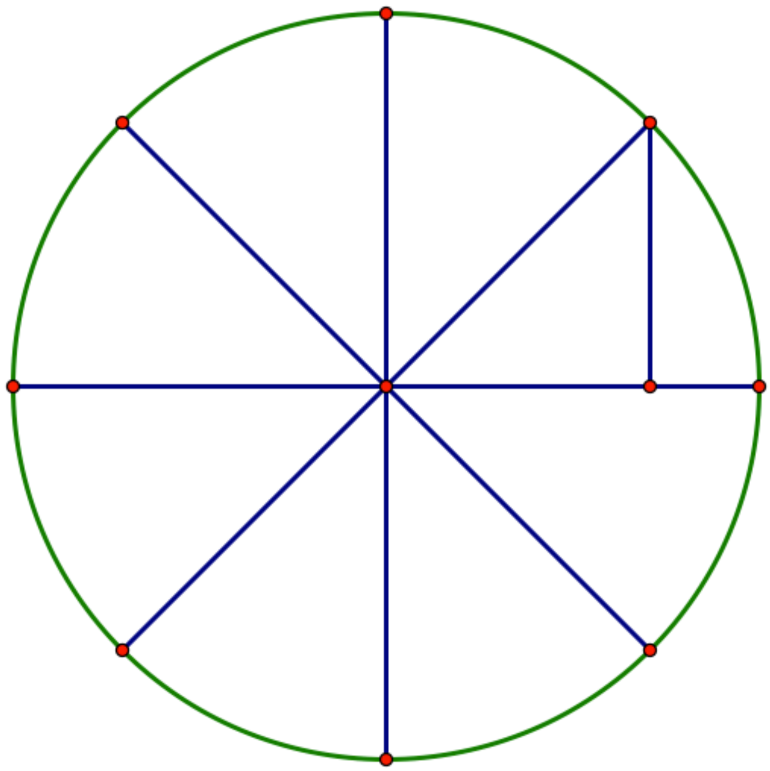
3. Use the diagram above to give exact values for the following trigonometric expressions.

a. $\sin\left(\frac{\pi}{6}\right) =$ b. $\sin\left(\frac{5\pi}{6}\right) =$ c. $\cos\left(\frac{7\pi}{6}\right) =$

d. $\sin\left(\frac{\pi}{3}\right) =$ e. $\cos\left(\frac{\pi}{6}\right) =$ f. $\cos\left(\frac{11\pi}{6}\right) =$

g. $\sin\left(\frac{3\pi}{2}\right) =$ h. $\cos(\pi) =$ i. $\sin\left(\frac{7\pi}{3}\right) =$

4. Here is a plan for an alternative water wheel with only 8 spokes. Label the exact values of the x and y-coordinates for each point on the following schematic drawing of the water wheel. (Hint: You might want to begin this work by finding the length of the "brace" shown in the diagram.)



5. Use the diagram above to give exact values for the following trigonometric expressions.

a. $\sin\left(\frac{\pi}{4}\right) =$

b. $\sin\left(\frac{5\pi}{4}\right) =$

c. $\cos\left(\frac{3\pi}{4}\right) =$

d. $\cos\left(\frac{\pi}{4}\right) =$

e. $\cos\left(-\frac{\pi}{4}\right) =$

f. $\sin\left(\frac{7\pi}{4}\right) =$

g. $\sin\left(\frac{3\pi}{2}\right) =$

h. $\cos\left(\frac{3\pi}{2}\right) =$

i. $\sin\left(\frac{11\pi}{4}\right) =$

During the spring runoff of melting snow the stream of water powering this water wheel causes it to make one complete revolution counterclockwise every 3 seconds.

6. Write an equation to represent the height of a particular paddle of the water wheel above or below the water level at any time *t* after the paddle emerges from the water.

   - Write your equation so the height of the paddle will be graphed correctly on a calculator set in *degree* mode.

   - Revise your equation so the height of the paddle will be graphed correctly on a calculator set in *radian* mode.

During the summer months the stream of water powering this water wheel becomes a "lazy river" causing the wheel to make one complete revolution counterclockwise every 12 seconds.

7. Write an equation to represent the height of a particular paddle of the water wheel above or below the water level at any time *t* after the paddle emerges from the water.

   - Write your equation so the height of the paddle will be graphed correctly on a calculator set in *degree* mode.

   - Revise your equation so the height of the paddle will be graphed correctly on a calculator set in *radian* mode.

6.9 Water Wheels and the Unit Circle – Teacher Notes

A Practice Understanding Task

Purpose: The purpose of this task is to practice finding the sine and cosine of angles of rotation on the unit circle when the angle is measured in radians counterclockwise from the ray passing through the point (1, 0); that is, angles drawn in standard position. The angles of rotation in this task are multiples of $\pi/6$ or $\pi/4$ so the trigonometric values can be expressed exactly—rather than as decimal approximations—by using properties of special right triangles.

Core Standards Focus:

F.TF.1 Understand radian measure of an angle as the length of the arc on the unit circle subtended by the angle.

F.TF.2 Explain how the unit circle in the coordinate plane enables the extension of trigonometric functions to all real numbers, interpreted as radian measures of angles traversed counterclockwise around the unit circle.

F.TF.3 (+) Use special triangles to determine geometrically the values of sine, cosine, tangent for $\pi/3$, $\pi/4$ and $\pi/6$, and use the unit circle to express the values of sine, cosine, and tangent for $\pi-x$, $\pi+x$, and $2\pi-x$ in terms of their values for x, where x is any real number.

Standards for Mathematical Practice:

SMP 5 – Use appropriate tools strategically

Vocabulary: In this task students will continue working with angles of rotation drawn in standard position on the *unit circle*, the circle centered at the origin with a radius of 1 unit. All of the angles of rotation in this task are multiples of $\pi/6$, $\pi/3$, and $\pi/4$.

The Teaching Cycle:

Launch (Whole Class):

Introduce the water wheel context, emphasizing that this wheel has a radius of one meter. Ask students to complete question 1 on the first page of the task. Students should be able to use the special relationships that exist between the side lengths of 30°-60°-90° triangles and 45°-45°-90° triangles in this task. These relationships were developed in the task *Special Rights* (see MVP Secondary Math III, module 5). You may need to remind students of these relationships, or allow students to rediscover these relationships for themselves. Once students agree on the coordinates of point B in each diagram, set them to work on the rest of the task.

Explore (Small Group):

Watch for how students label all of the points on each unit circle. Do they make use of the symmetry of the circle in their labeling? Do they label points to the left of the center with negative x-coordinates and points below the center with negative y-coordinates?

Listen for how students use the unit circle diagram to fill out the tables. Do they correctly identify the x-coordinate as the *cosine* of the angle of rotation whose terminal ray passes through that point, and the y-coordinate as the *sine* of that angle?

The last page of the task gives students an opportunity to review writing equations to model circular motion, and to consider how to revise their equations in terms of radian measurement of angles of rotation. Allow students time to work on these problems, and select students to present who can clearly explain how to revise these equations in terms of radians.

Discuss (Whole Class):

Post a unit circle with points plotted every $^{\pi}/_{6}$ radians and one with points every $^{\pi}/_{4}$ radians. Label only the points in the first quadrant with coordinates. Ask students to find values of the sine and cosine for various angles that are multiples of $^{\pi}/_{6}$ and $^{\pi}/_{4}$ using only the labeled points in quadrant I to assist their thinking.

Discuss the functions students have written on the last page of the task. Have students graph their equations in both degree and radian mode to check the accuracy of their revised equations.

Aligned Ready, Set, Go: Modeling Periodic Behavior 6.9

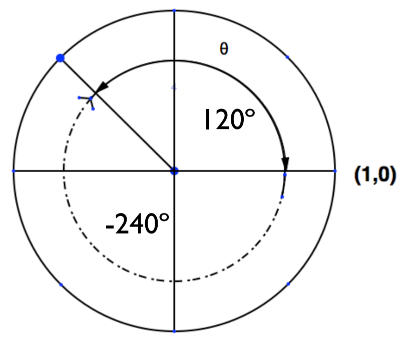
READY, SET, GO! Name Period Date

READY

Topic: Identifying coterminal angles

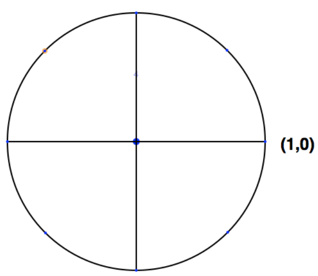
State a negative angle of rotation that is *coterminal* with the given angle of rotation. (*Coterminal* angles share the same terminal side of an angle of rotation.) Sketch and label both angles.

Example: $\theta = 120°$ is the given angle of rotation. The angle of rotation is indicated by the solid arc. The dotted angle of rotation is a coterminal angle with a rotation of $-240°$.



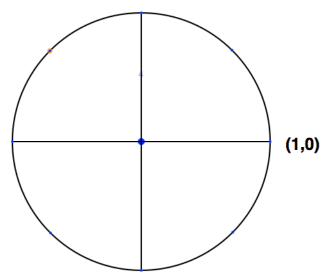
1. Given $\theta = 20°$

Coterminal angle ______________________



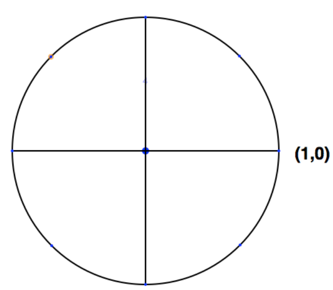
2. Given $\theta = 95°$

Coterminal angle ______________________



3. Given $\theta = 225°$

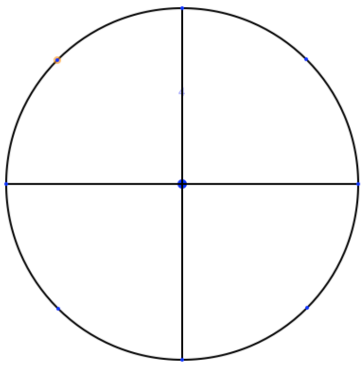
Coterminal angle ______________________



mathematics vision project

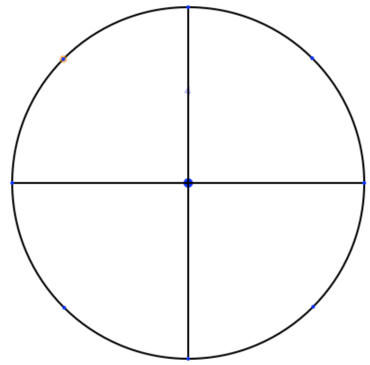
4. Given $\theta = 270°$

Coterminal angle ______________________



5. Given $\theta = 300°$

Coterminal angle ______________________



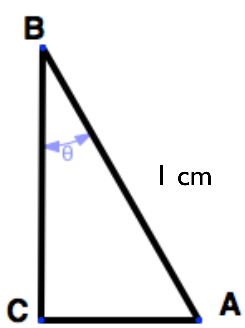
6. What is the sum of a positive angle of rotation and the absolute value of its negative coterminal angle?

7. Every angle has an infinite number of coterminal angles both positive and negative if the definition is extended to angles of rotation greater than 360°. For example: an angle of 45° is coterminal with angles of rotation measuring 405°, 765° *etc*. Given $\theta = 115°$, name 3 **positive** coterminal angles.

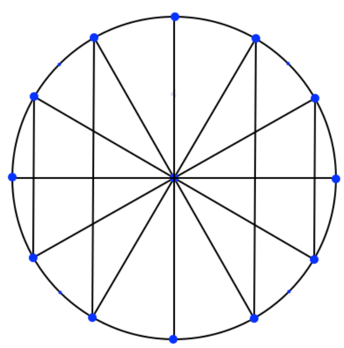
SET

Topic: Calculating sine and cosine of radian measures

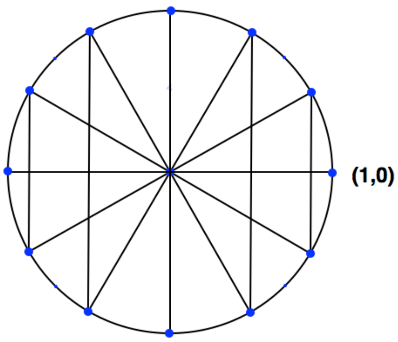
8. Triangle ABC is a 30°, 60°, 90° right triangle. The length of one side is given. Fill in the values for the missing sides. $m\angle B = 30°$.



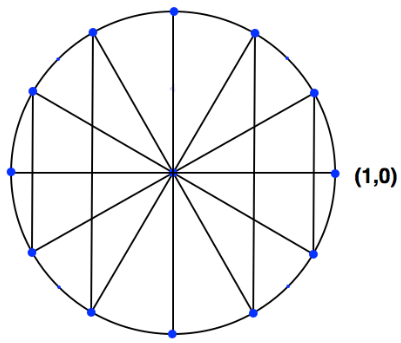
9. Label each point around the circle with the angle of rotation in radians starting from the point (1,0).



10. Use the values in #8 to write the **exact** coordinates of the points on the circle below. Be mindful of the numbers that are negative.



11. Find the arc length, *s*, from the point (1,0) to each point around the circle. Record your answers as decimal approximations to the nearest thousandth.



Use your calculator to find the following values.

12. $\sin\frac{5\pi}{6} =$

13. $\sin\frac{\pi}{3} =$

14. Why are both of your answers positive?

15. $\cos\frac{2\pi}{3} =$

16. $\cos\frac{4\pi}{3} =$

17. Why are both of your answers negative?

18. $\sin\frac{\pi}{2} =$

19. $\cos\frac{\pi}{2} =$

20. In which quadrants are sine and cosine both negative?

21. Name an angle of rotation where sine is equal to -1.

22. Name an angle of rotation where cosine is equal to -1.

GO

Topic: Finding the angle when given the trig ratio

Use your calculator to find the value of $\boldsymbol{\theta}$ *where* $\mathbf{0 \leq \theta \leq 90°}$. Round your answers to the nearest degree.

23. $\sin\theta = 0.82$

24. $\cos\theta = 0.31$

25. $\cos\theta = 0.98$

26. $\sin\theta = 0.39$

27. $\sin\theta = 1$

28. $\cos\theta = 0.71$