mathematics vision project

Transforming Mathematics Education

ALGEBRA II

An Integrated Approach

MODULE 2

Logarithmic Functions

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The Mathematics Vision Project

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MODULE 2 - TABLE OF CONTENTS

LOGARITHMIC FUNCTIONS

2.1 Log Logic

A Develop Understanding Task

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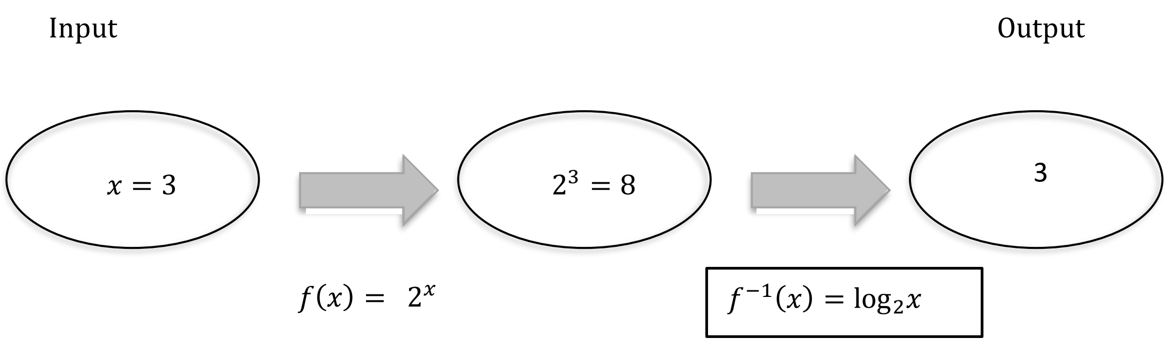
We began thinking about logarithms as inverse functions for exponentials in *Tracking the Tortoise*. Logarithmic functions are interesting and useful on their own. In the next few tasks, we will be working on understanding logarithmic expressions, logarithmic functions, and logarithmic operations on equations.

We showed the inverse relationship between exponential and logarithmic functions using a diagram like the one below:



We could summarize this relationship by saying:

$2^3 = 8$ so, $\log_2 8 = 3$

Logarithms can be defined for any base used for an exponential function. Base 10 is popular. Using base 10, you can write statements like these:

$10^1 = 10$ so, $\log_{10} 10 = 1$

$10^2 = 100$ so, $\log_{10} 100 = 2$

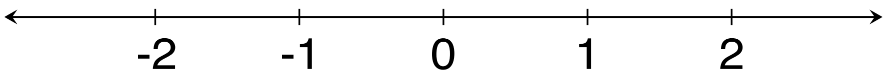
$10^3 = 1000$ so, $\log_{10} 1000 = 3$

The notation may see different, but you can see the inverse pattern where the inputs and outputs switch.

The next few problems will give you an opportunity to practice thinking about this pattern and possibly make a few conjectures about other patterns related to logarithms.

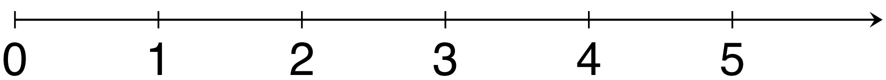
Place the following expressions on the number line. Use the space below the number line to explain how you knew where to place each expression.

1. A. $\log_3 3$ B. $\log_3 9$ C. $\log_3\left(\frac{1}{3}\right)$ D. $\log_3 1$ E. $\log_3\left(\frac{1}{9}\right)$



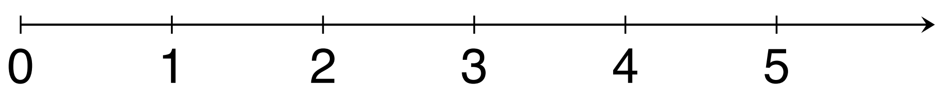
Explain: ____

2. A. $\log_3 81$ B. $\log_{10} 100$ C. $\log_8 8$ D. $\log_5 25$ E. $\log_2 32$



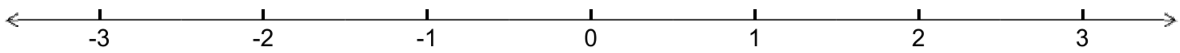
Explain: ____

3. A. $\log_7 7$ B. $\log_9 9$ C. $\log_{11} 1$ D. $\log_{10} 1$



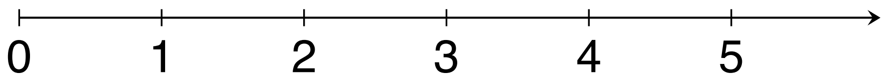
Explain: ____

4. A. $\log_2\left(\frac{1}{4}\right)$ B. $\log_{10}\left(\frac{1}{1000}\right)$ C. $\log_5\left(\frac{1}{125}\right)$ D. $\log_6\left(\frac{1}{6}\right)$



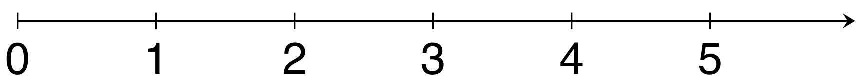
Explain: ____

5. A. $\log_4 16$ B. $\log_2 16$ C. $\log_8 16$ D. $\log_{16} 16$



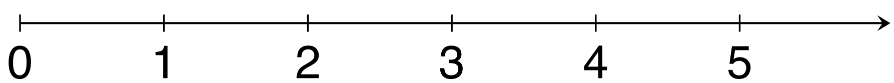
Explain: ____

6. A. $\log_2 5$ B. $\log_5 10$ C. $\log_6 1$ D. $\log_5 5$ E. $\log_{10} 5$



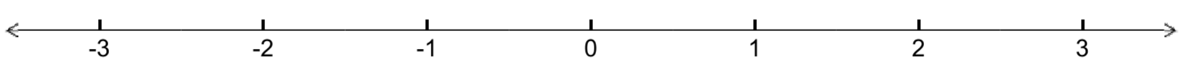
Explain: ____

7. A. $\log_{10} 50$ B. $\log_{10} 150$ C. $\log_{10} 1000$ D. $\log_{10} 500$



Explain: ____

8. A. $\log_3 3^2$ B. $\log_5 5^{-2}$ C. $\log_6 6^0$ D. $\log_4 4^{-1}$ E. $\log_2 2^3$



Explain: ____

Based on your work with logarithmic expressions, determine whether each of these statements is always true, sometimes true, or never true. If the statement is sometimes true, describe the conditions that make it true. Explain your answers.

9. The value of $\log_b x$ is positive.

Explain: ____

10. $\log_b x$ is not a valid expression if x is a negative number.

Explain: ____

11. $\log_b 1 = 0$ for any base, b > 0.

Explain: ____

12. $\log_b b = 1$ for any b > 0.

Explain: ____

13. $\log_2 x < \log_3 x$ for any value of x.

Explain: ____

14. $\log_{\mathrm{b}} b^n = n$ for any b > 0.

Explain: ____

2.1 Log Logic – Teacher Notes

A Develop Understanding Task

Purpose:

The purpose of this task is to develop students' understanding of logarithmic expressions and to make sense of the notation. In addition to evaluating log expressions, student will compare expressions that they cannot evaluate explicitly. They will also use patterns they have seen in the task and the definition of a logarithm to justify some properties of logarithms.

Core Standards Focus:

F.BF.5 (+) Understand the inverse relationship between exponents and logarithms and use this relationship to solve problems involving logarithms and exponents.

F.LE.4 For exponential models, express as a logarithm the solution to $ab^{ct} = d$ where a, c, and d are numbers and the base b is 2, 10, or e; evaluate the logarithm using technology.

Note for F.LE.4: *Consider extending this unit to include the relationship between properties of logarithms and properties of exponents, such as the connection between the properties of exponents and the basic logarithm property that log xy = log x + log y.*

Related Standards: F.BF.4

Standards for Mathematical Practice:

SMP 2 – Reason abstractly and quantitatively

SMP 8 – Look for and express regularity in repeated reasoning

Vocabulary: base of a logarithm, argument of a logarithm

The Teaching Cycle:

Launch (Whole Class):

Begin by working through each of the examples on page 1 of the task with students. Tell them that since we know that logarithmic functions and exponential functions are inverses, the definition of a logarithm is:

If $b^x = n$ then $\log_b n = x$ for $b > 0$

Keep this relationship posted where students can refer to it during their work on the task.

Explore (Small Group):

The task begins with expressions that will generate integer values. In the beginning, encourage students to use the pattern expressed in the definition to help find the values. If they don't know the powers of the base numbers, they may need to use calculators to identify them. For instance, if they are asked to evaluate $\log_2 32$, they may need to use the calculator to find $2^5 = 32$. (Author's note: I hope this won't be the case, but the emphasis in this task is on reasoning, not on arithmetic skill.) Thinking about these values will help to review integer exponents.

Starting at #5, there are expressions that can only be estimated and placed on the number line in a reasonable location. Don't give students a way to use the calculator to evaluate these expressions directly; again the emphasis is on reasoning and comparing.

As you monitor students as they work, keep track of students that have interesting justifications for their answers on problems #9 – 15 so that they can be included in the class discussion.

Discuss (Whole Class):

Begin the discussion with #2. For each log expression, write the equivalent exponential equation like so:

$\log_3 81 = 4 \quad 3^4 = 81$

This will give students practice in seeing the relationship between exponential functions and logarithmic functions. Place each of the values on the number line.

Move the discussion to #4 and proceed in the same way, giving students a brush-up on negative exponents.

Next, work with question #5. Since students can't calculate all of these expressions directly, they will have to use logic to order the expressions. One strategy is to first put the expressions in order from smallest to biggest based on the idea that the bigger the base, the smaller the exponent will need to be to get 16. (Be sure this idea is generalized by the end of the discussion of #5.) Once the numbers are in order, then the approximate values can be considered based upon known values for a particular base.

Work on #7 next. In this problem, the bases are the same, but the arguments are different. The expressions can be ordered based on the idea that for a given base, $b > 1$, the greater the argument, the greater the exponent will need to be.

Finally, work through each of problems 9 – 15. This is an opportunity to develop a number of the properties of logarithms from the definitions. After students have justified each of the properties that are always true (#10, 11, 12, and 14), these should be posted in the classroom as agreed-upon properties that can be used in future work.

Aligned Ready, Set, Go: *Logarithmic Functions 2.1*

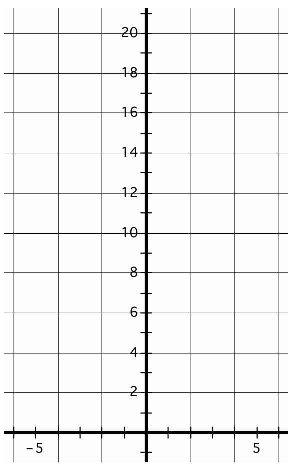
READY, SET, GO! Name Period Date

READY

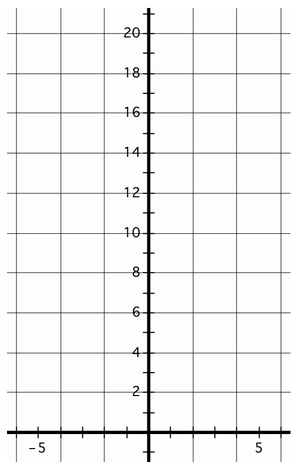
Topic: Graphing exponential equations

Graph each function over the domain $\{-4 \leq x \leq 4\}$.

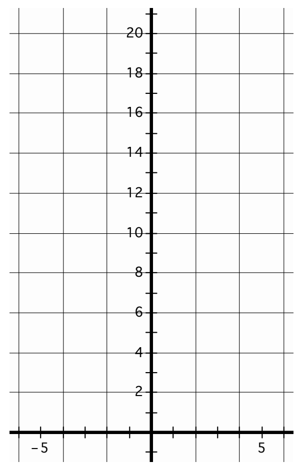
1. $y = 2^x$



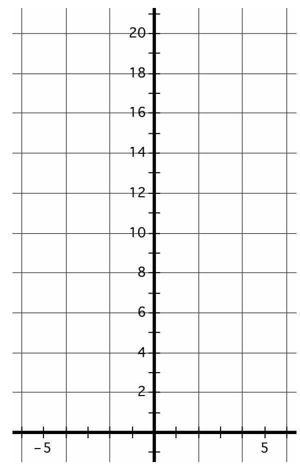
2. $y = 2 \cdot 2^x$



3. $y = \left(\frac{1}{2}\right)^x$



4. $y = 2\left(\frac{1}{2}\right)^x$



5. Compare graph #1 to graph #2. Multiplying by 2 should generate a dilation of the graph, but the graph looks like it has been translated vertically. How do you explain that?

6. Compare graph #3 to graph #4. Is your explanation in #5 still valid for these two graphs? Explain.

SET

Topic: Writing the logarithmic form of an exponential equation.

Definition of Logarithm: *For all positive numbers a, where a ≠ 1, and all positive numbers x,*

$$y = \log_a x \text{ means the same as } x = a^y.$$

*(Note the **base** of the exponent and the **base** of the logarithm are both **a**.)*

7. Why is it important that in the definition of logarithm it is stated that the base of the logarithm does not equal 1?

8. Why is it important that the definition of a logarithm states that the base of the logarithm is positive?

9. Why is it necessary that the definition states that $\boldsymbol{x}$ in the expression $\boldsymbol{log_a x}$ is positive?

Write the following exponential equations in logarithmic form.

Exponential form	**Logarithmic form**	**Exponential form**	**Logarithmic form**
10. $5^4 = 625$		11. $3^2 = 9$	
12. $\left(\frac{1}{2}\right)^{-3} = 8$		13. $4^{-2} = \frac{1}{16}$	
14. $10^4 = 10000$		15. $a^y = x$	

16. Compare the exponential form of an equation to the logarithmic form of an equation. What part of the exponential equation is the **answer** to the logarithmic equation?

Topic: Considering values of logarithmic functions

Answer the following questions. If yes, give an example of the answer. If no, explain why not.

17. Is it possible for a logarithm to equal a negative number?

18. Is it possible for a logarithm to equal zero?

19. Does $log_x 0$ have an answer?

20. Does $log_x 1$ have an answer?

21. Does $log_x x^5$ have an answer?

GO

Topic: Reviewing properties of Exponents

Write each expression as an integer or a simple fraction.

22. 27^0

23. $11(-6)^0$

24. -3^{-2}

25. 4^{-3}

26. $\frac{9}{2^{-1}}$

27. $\frac{4^3}{8^0}$

28. $3\left(\frac{29^3}{11^5}\right)^0$

29. $\frac{3}{6^{-1}}$

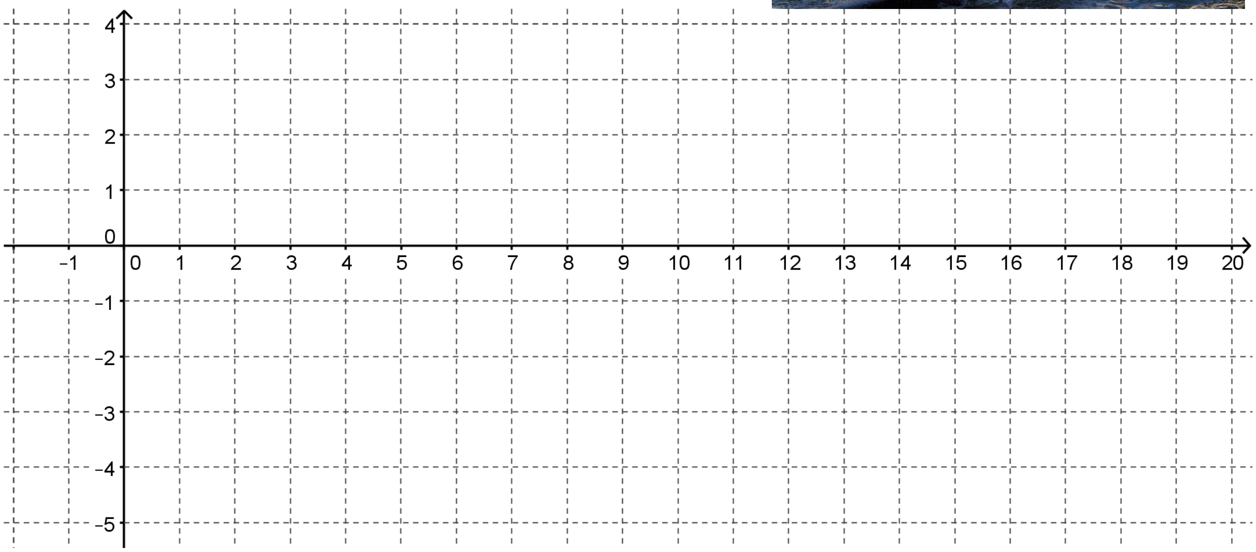
30. $\frac{32^{-1}}{4^{-1}}$

2.2 Falling Off a Log

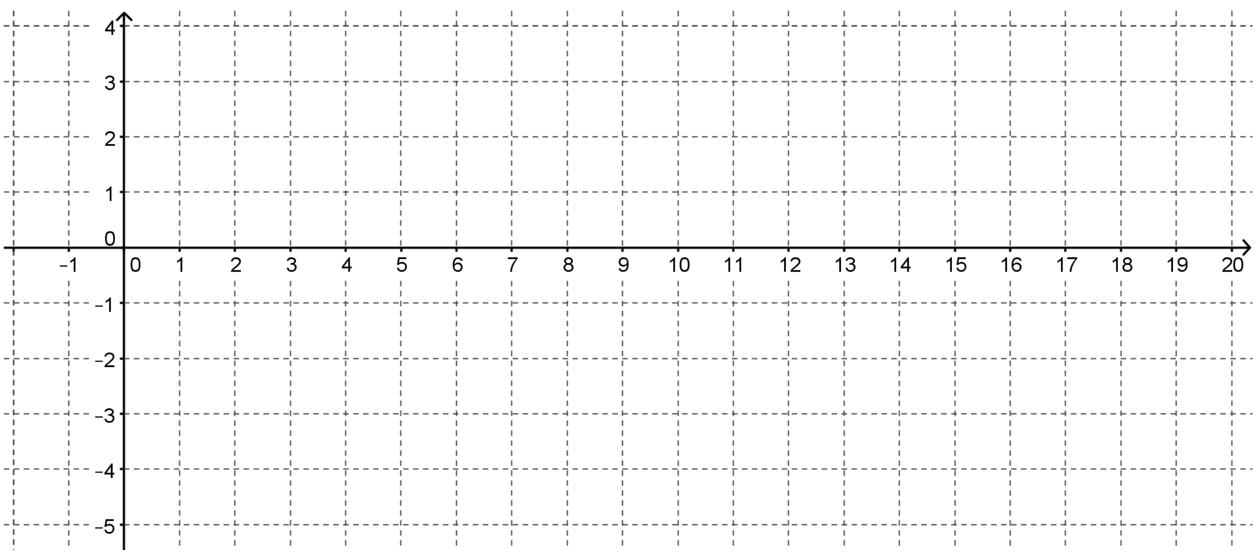
A Solidify Understanding Task

1. Construct a table of values and a graph for each of the following functions. Be sure to select at least two values in the interval $0 < x < 1$.

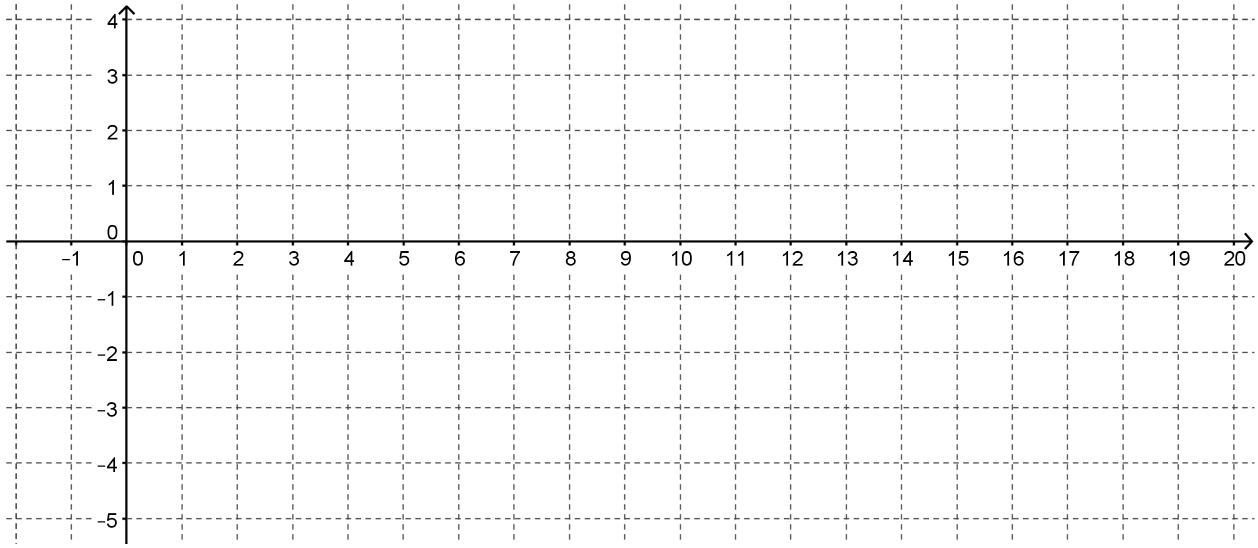
a) $f(x) = \log_2 x$



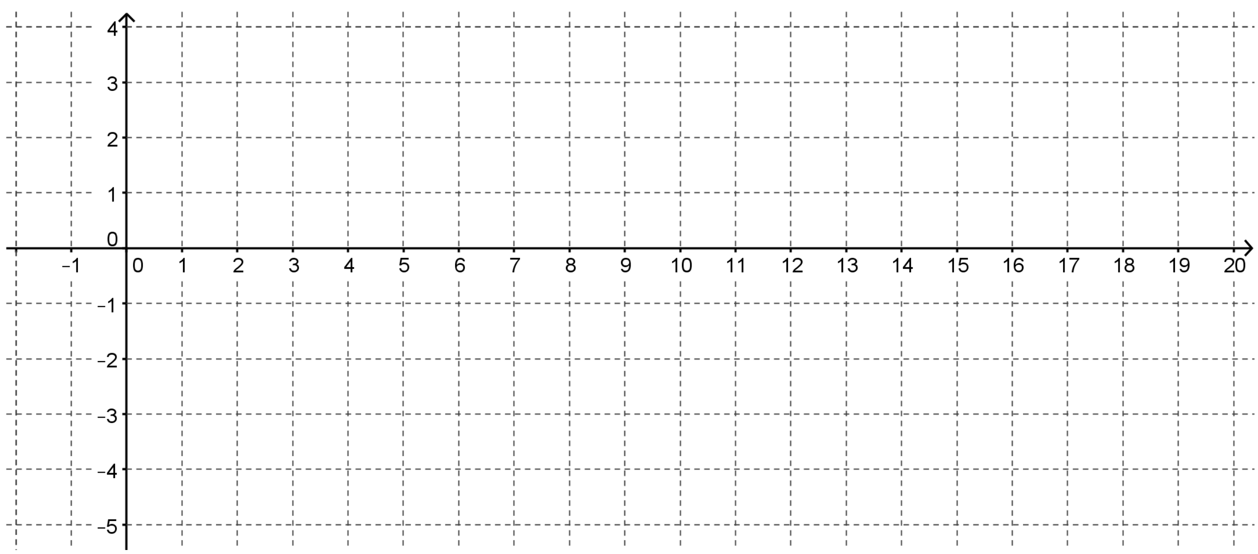
b) $g(x) = \log_3 x$



c) $h(x) = \log_4 x$



d) $k(x) = \log_{10} x$



2. How did you decide what values to use for x in the table?

3. How did you use the x values to find the y values in the table?

4. What similarities do you see in the graphs?

5. What differences do you observe in the graphs?

6. What is the effect of changing the base on the graph of a logarithmic function?

Let's focus now on $k(x) = \log_{10} x$ so that we can use technology to observe the effects of changing parameters on the function. **Because base 10 is a commonly used base for exponential and logarithmic functions, it is often abbreviated and written without the base, like this: $\boldsymbol{k(x) = \log x}$.**

7. Use technology to graph $y = \log x$. How does the graph compare to the graph that you constructed?

8. How do you predict that the graph of $y = a + \log x$ will be different from the graph of $y = \log x$?

9. Test your prediction by graphing $y = a + \log x$ for various values of a. What is the effect of a on the graph? Make a general argument for why this would be true for all logarithmic functions.

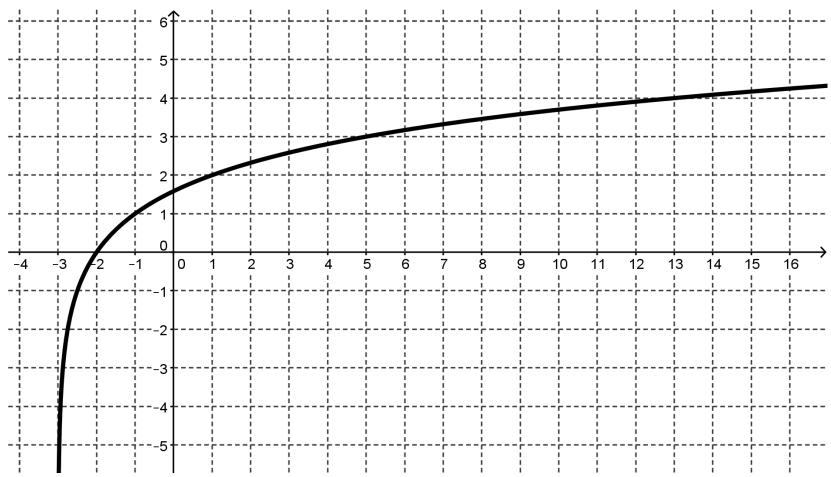
10. How do you predict that the graph of $y = \log(x + b)$ will be different from the graph of $y = \log x$?

11. Test your prediction by graphing $y = \log(x + b)$ for various values of b.
    - What is the effect of adding b?
    - What will be the effect of subtracting b (or adding a negative number)?

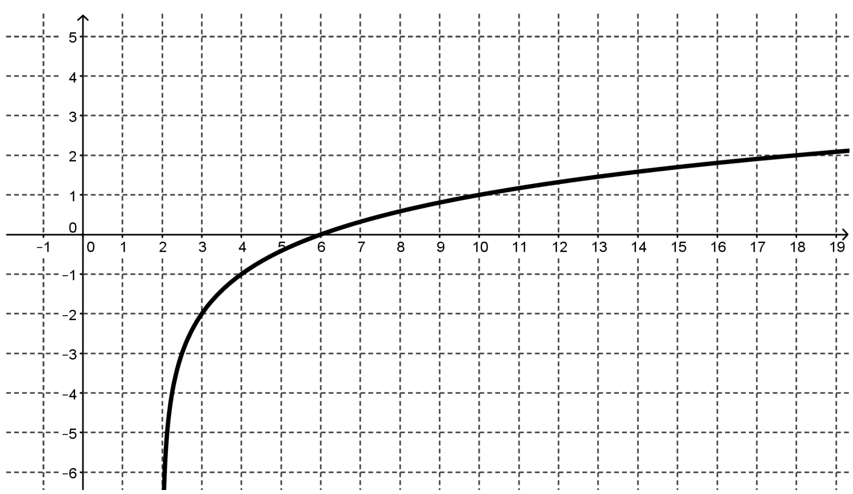
- Make a general argument for why this is true for all logarithmic functions.

12. Write an equation for each of the following functions that are transformations of $f(x) = \log_2 x$.

a)

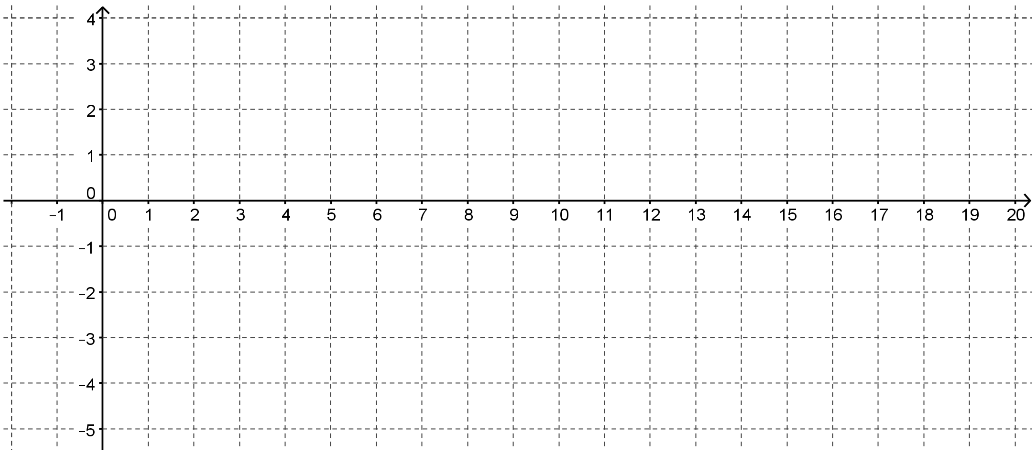


b)

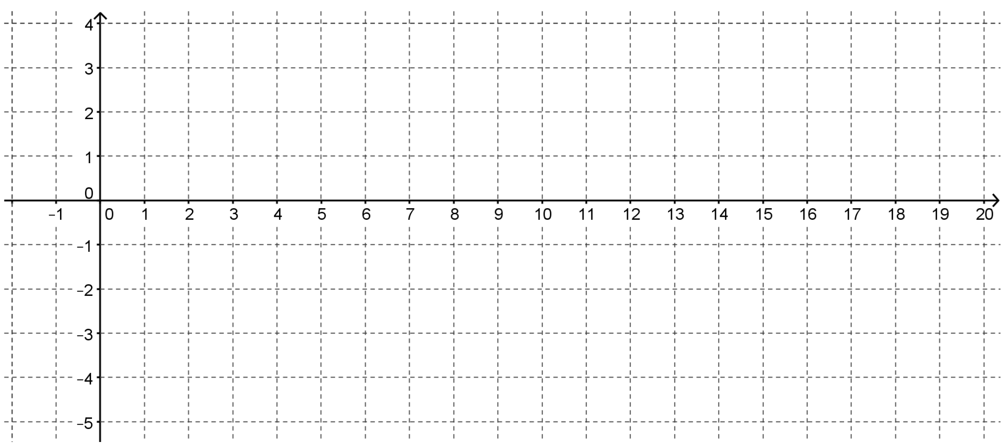


13. Graph and label each of the following functions:

a) $f(x) = 2 + \log_2(x-1)$



b) $g(x) = -1 + \log_2(x+2)$



14. Compare the transformation of the graphs of logarithmic functions with the transformation of the graphs of quadratic functions.

2. 2 Falling Off a Log – Teacher Notes

A Solidify Understanding Task

Purpose: The purpose of this task is to build on students' understanding of a logarithmic function as the inverse of an exponential function and their previous work in determining values for logarithmic expressions to find the graphs of logarithmic functions of various bases. Students use technology to explore transformations with log graphs in base 10 and then generalize the transformations to other bases.

Core Standards Focus:

F.IF.7.a Graph exponential and logarithmic functions, showing intercepts and end behavior, and trigonometric functions, showing period, midline, and amplitude.

F.BF.5 (+) Understand the inverse relationship between exponents and logarithms and use this relationship to solve problems involving logarithms and exponents.

Note for F. BF: Use transformations of functions to find more optimum models as students consider increasingly more complex situations.

For F.BF.3, note the effect of multiple transformations on a single function and the common effect of each transformation across function types. Include functions defined only by a graph.

Extend F.BF.4a to simple rational, simple radical, and simple exponential functions; connect F.BF.4a to F.LE.4.

Related Standards: F.LE.4

Standards for Mathematical Practice:

SMP 2 – Reason abstractly and quantitatively

SMP 5 – Use appropriate tools strategically

Vocabulary: vertical asymptote

Note to Teachers: Access to graphing technology is necessary for this task.

A Desmos activity for this is task is available at: https://tinyurl.com/MVPMath3Lesson2-2. It can be found at Desmos by searching for: *MVP Math III, 2.2 Falling Off a Log*. The activity takes students through questions 7-11, allowing them to use technology to explore the vertical and horizontal shifts on the graphs of a logarithmic function.

The Teaching Cycle

Launch (Whole Class):

Begin class by reminding students of the work they did with log expressions in the previous task and soliciting a few exponential and log statements like this:

$$5^3 = 125 \quad so \quad \log_5 125 = 3$$

Encourage the use of different bases to remind students that the same definition works for all bases, b > 1. Tell students that in this task they will use what they know about inverses to help them create tables and graph log functions.

Explore (Small Group):

Monitor students as they work to ensure they are completing both tables and graphs for each function. Some students may choose to graph the exponential function and then reflect it over the $y = x$ line to get the graph before completing the table. Watch for this strategy and be prepared to highlight it during the discussion. Finding points on the graph for $0 < x < 1$ may prove difficult for students since negative exponents are often difficult. If students are stuck, remind them that it may be easier to find points on the exponential function and then switch them for the log graphs.

Discuss (Whole Class):

Begin the discussion with the graph of $f(x) = \log_2 x$. Ask a student that used the exponential function $y = 2^x$ and switched the x and y values to present their graph. Then have a student that started by creating a table describe how they obtained the values in the table. Ask the class to identify how the two strategies are connected. By now, students should be able to articulate the idea that powers of 2 are easy values to think about and the value of the log expression will be the exponent in each case.

Move the discussion to question #4, the similarities between the graphs. Students will probably speak generally about the shapes being alike. In the discussions of similarities, be sure that the more technical features of the graphs emerge:

- The point (1,0) is included
- The domain is (0, ∞)
- The range is (-∞,∞)
- The function is increasing over the entire domain.

Ask students to connect each of these features with the definition of a logarithm and properties of inverse functions. At this point, introduce the idea of a vertical asymptote. We know that logarithmic functions are not defined for $x = 0$ because exponential functions approach, but never actually reach 0. Since logarithmic functions are inverses of exponential functions, we would expect their graphs to be reflections over the $y = x$ line. Also help students to understand why the y-values of a logarithmic function become very large negative numbers as x-values become closer to 0.

Ask what conclusions they could draw about the effect of changing the base on graph. How do these conclusions connect to the strategies they used to order log expressions with different bases in the previous task?

Ask students how the graphs were transformed when a number is added outside the log functions versus inside the argument of the log function. Students should notice that this is just like other functions that they are familiar with such as quadratic functions.

Aligned Ready, Set, Go: *Logarithmic Functions 2.2*

READY, SET, GO! Name Period Date

READY

Topic: Solving simple logarithmic equations

Find the answer to each logarithmic equation. Then explain how each equation supports the statement, *"The answer to a logarithmic equation is always the exponent."*

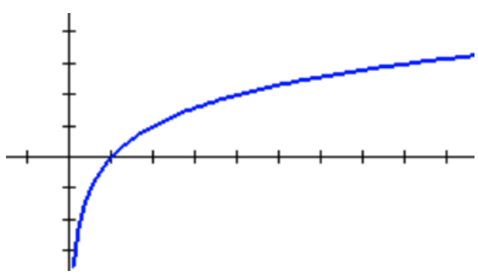
1. $log_5 625 =$
2. $log_3 243 =$
3. $log_5 0.2 =$
4. $log_9 81 =$
5. $log 1{,}000{,}000 =$
6. $log_x x^7 =$

SET

Topic: Exploring transformations on logarithmic functions

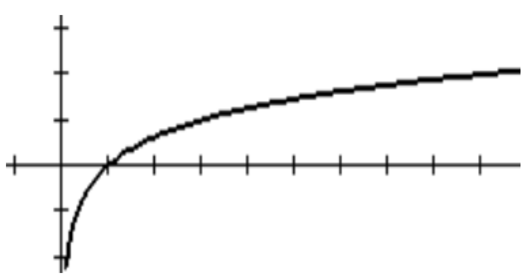
Answer the questions about each graph.

7.



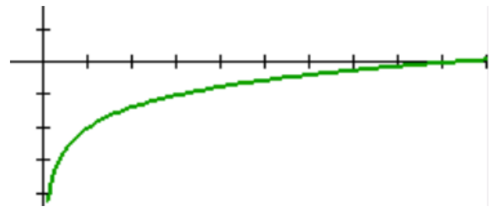
a. What is the value of x when $f(x) = 0$?
b. What is the value of x when $f(x) = 1$?
c. What is the value of $f(x)$ when $x = 2$?
d. What will be the value of x when $f(x) = 4$?
e. What is the equation of this graph?

8.



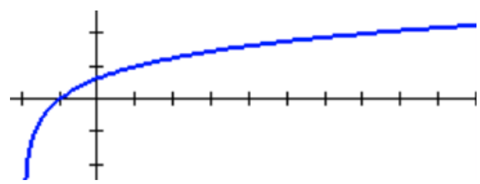
a. What is the value of x when $f(x) = 0$?
b. What is the value of x when $f(x) = 1$?
c. What is the value of $f(x)$ when $x = 9$?
d. What will be the value of x when $f(x) = 4$?
e. What is the equation of this graph?

9. Use the graph and the table of values for the graph to write the equation of the graph. Explain which numbers in the table helped you the most to write the equation.



X	Y_1
1	-2
2	-1.369
3	-1
5	-.535
7	-.2288
9	0

10. Use the graph and the table of values for the graph to write the equation of the graph. Explain which numbers in the table helped you the most to write the equation.



X	Y_1
-2	ERROR
-1	0
0	.63093
1	1
5	1.7712
6	1.8928
7	2

GO

Topic: Using the power to a power rule with exponents

Simplify each expression. Answers should have only positive exponents.

11. $(2^3)^4$
12. $(x^3)^2$
13. $(a^3)^{-2}$
14. $(2^3w)^4$
15. $(b^{-7})^3$
16. $(d^{-3})^{-2}$
17. $x^2 \cdot (x^5)^2$
18. $m^{-3} \cdot (m^2)^3$
19. $(x^5)^{-4} \cdot x^{25}$
20. $(5a^3)^2$
21. $(6^{-3})^2$
22. $(2a^3b^2)^2$

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2.3 Chopping Logs
A Solidify Understanding Task

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Abe and Mary were working on their math homework together when Abe has a brilliant idea!

Abe: I was just looking at this *log* function that we graphed in *Falling Off A Log*:

$$y = \log_2(x+b).$$

I started to think that maybe I could just "distribute" the *log* so that I get:

$$y = \log_2 x + \log_2 b.$$

I guess I'm saying that I think these are equivalent expressions, so I could write it this way:

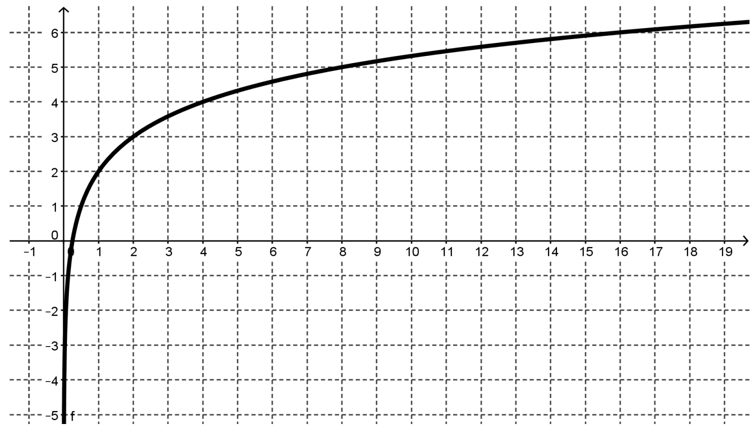
$$\log_2(x+b) = \log_2 x + \log_2 b$$

Mary: I don't know about that. *Logs* are tricky and I don't think that you're really doing the same thing here as when you distribute a number.

1. What do you think? How can you verify if Abe's idea works?

2. If Abe's idea works, give some examples that illustrate why it works. If Abe's idea doesn't work, give a counter-example.

Abe: I just know there is something going on with these *logs*. I just graphed $f(x) = \log_2(4x)$. Here it is:



It's weird because I think this graph is just a translation of $y = \log_2 x$. Is it possible the equation of this graph could be written more than one way?

3. How would you answer Abe's question? Are there conditions that could allow the same graph to have different equations?

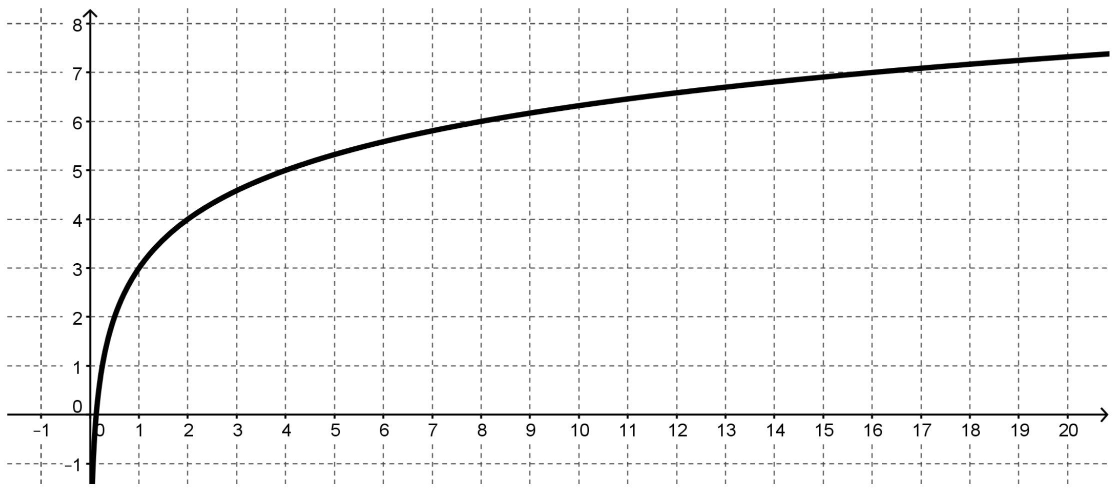
Mary: When you say, "a translation of $y = \log_2 x$" do you mean that it is just a vertical or horizontal shift? What could that equation be?

4. Find an equation for $f(x)$ that shows it to be a horizontal or vertical shift of $y = \log_2 x$.

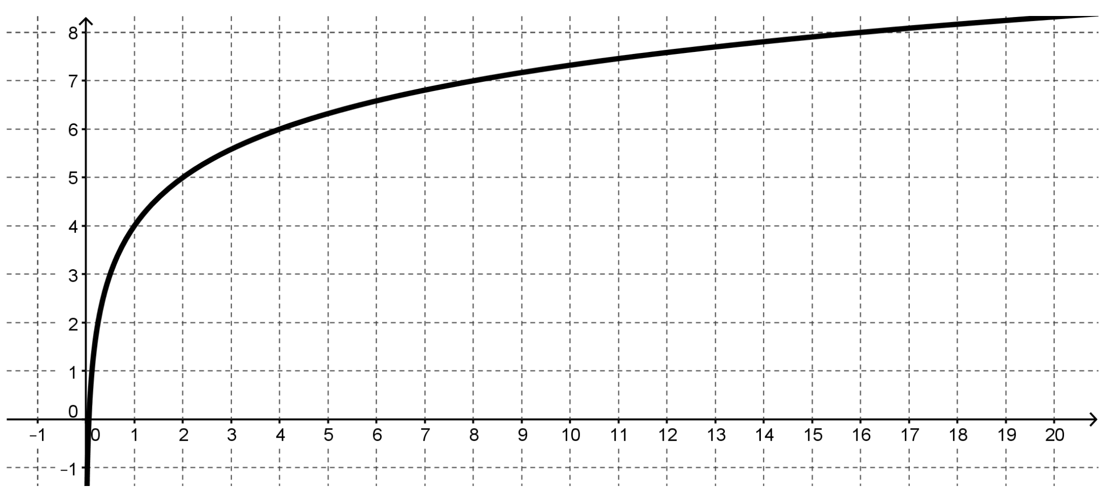
Mary: I wonder why the vertical shift turned out to be up 2 when the x was multiplied by 4. I wonder if it has something to do with the power that the base is raised to, since this is a *log* function. Let's try to see what happens with $y = \log_2(8x)$ and $y = \log_2(16x)$.

5. Try to write an equivalent equation for each of these graphs that is a vertical shift of $y = \log_2 x$.

a) $y = \log_2(8x)$ Equivalent equation: ______________________________



b) $y = \log_2(16x)$ Equivalent equation: ______________________________



Mary: Oh my gosh! I think I know what is happening here! Here's what we see from the graphs:

$$\log_2(4x) = 2 + \log_2 x$$

$$\log_2(8x) = 3 + \log_2 x$$

$$\log_2(16x) = 4 + \log_2 x$$

Here's the brilliant part: We know that $\log_2 4 = 2, \log_2 8 = 3,$ and $\log_2 16 = 4$. So:

$$\log_2(4x) = \log_2 4 + \log_2 x$$

$$\log_2(8x) = \log_2 8 + \log_2 x$$

$$\log_2(16x) = \log_2 16 + \log_2 x$$

I think it looks like the "distributive" thing that you were trying to do, but since you can't really distribute a function, it's really just a *log* multiplication rule. I guess my rule would be:

$$\log_2(ab) = \log_2 a + \log_2 b$$

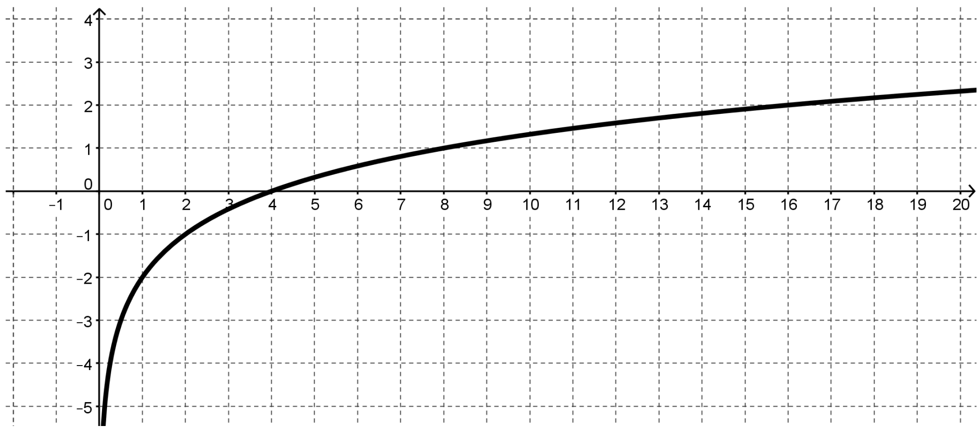
6. How can you express Mary's rule in words?

7. Is this statement true? If it is, give some examples that illustrate why it works. If it is not true provide a counter-example.

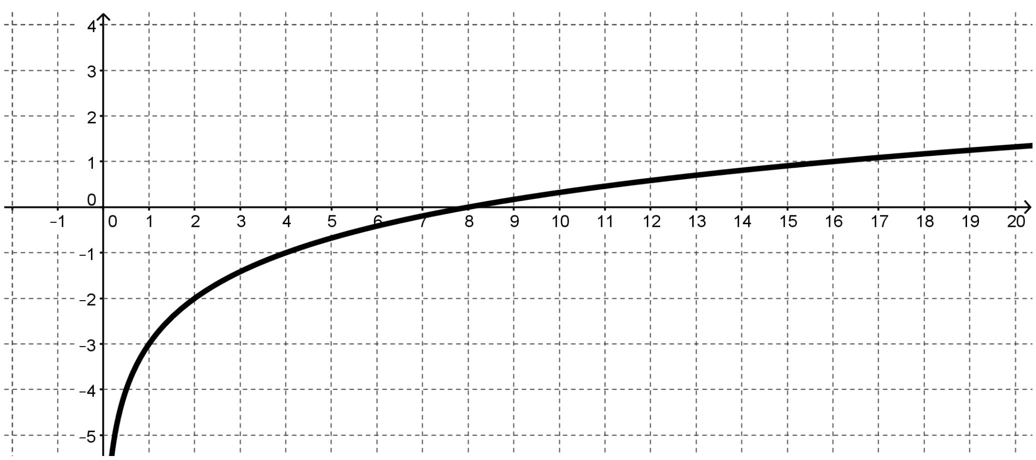
Mary: So, I wonder if a similar thing happens if you have division inside the argument of a *log* function. I'm going to try some examples. If my theory works, then all of these graphs will just be vertical shifts of $y = \log_2 x$.

8. Here are Abe's examples and their graphs. Test Abe's theory by trying to write an equivalent equation for each of these graphs that is a vertical shift of $y = \log_2 x$.

a) $y = \log_2\left(\frac{x}{4}\right)$ Equivalent equation: ______________________



b) $y = \log_2\left(\frac{x}{8}\right)$ Equivalent equation: ______________________



9. Use these examples to write a rule for division inside the argument of a *log* that is like the rule that Mary wrote for multiplication inside a *log*.

10. Is this statement true? If it is, give some examples that illustrate why it works. If it is not true provide a counter-example.

Abe: You're definitely brilliant for thinking of that multiplication rule. But I'm a genius because I've used your multiplication rule to come up with a power rule. Let's say you start with:

$\log_2(x^3)$

Really that's the same as having: $\log_2(x \cdot x \cdot x)$

So, I could use your multiplying rule and write: $\log_2 x + \log_2 x + \log_2 x$

I notice there are 3 terms that are all the same. That makes it: $3\log_2 x$

So my rule is: $\log_2(x^3) = 3\log_2 x$

If your rule is true, then I have proven my power rule.

Mary: I don't think it's really a power rule unless it works for any power. You only showed how it might work for 3.

Abe: Oh, good grief! Ok, I'm going to say it can be any number x, raised to any power, k. My power rule is:

$$\log_2(x^k) = k\log_2 x$$

Are you satisfied?

11. Provide an argument about Abe's power rule. Is it true or not?

Abe: Before we win the Nobel Prize for mathematics I suppose that we need to think about whether or not these rules work for any base.

12. The three rules, written for any base $b > 1$ are:

    Log of a Product Rule: $\log_b(xy) = \log_b x + \log_b y$

    Log of a Quotient Rule: $\log_b\left(\frac{x}{y}\right) = \log_b x - \log_b y$

    Log of a Power Rule: $\log_b(x^k) = k\log_b x$

    Make an argument for why these rules will work in any base $b > 1$ if they work for base 2.

13. How are these rules similar to the rules for exponents? Why might exponents and *logs* have similar rules?

2.3 Chopping Logs – Teacher Notes

A Solidify Understanding Task

Note to Teachers: Access to graphing technology is necessary for this task.

Purpose: The purpose of this task is to use student understanding of log graphs and log expressions to derive properties of logarithms. In the task students are asked to find equivalent equations for graphs and then to generalize the patterns to establish the product, quotient, and power rules for logarithms.

Core Standards Focus:

F.IF.8. Write a function defined by an expression in different but equivalent forms to reveal and explain different properties of the function.

F.LE.4. For exponential models, express as a logarithm the solution to $ab^{ct} = d$ where a, c, and d are numbers and the base b is 2, 10, or e; evaluate the logarithm using technology.

Note to F.LE.4: *Consider extending this unit to include the relationship between properties of logarithms and properties of exponents, such as the connection between the properties of exponents and the basic logarithm property that log(xy) = log x + log y.*

Related Standards: F.BF.5

Standards for Mathematical Practice:

SMP 5 – Use appropriate tools strategically

SMP 7 – Look for and make use of structure

The Teaching Cycle:

Note: The nature of this task suggests a more guided approach from the teacher than many tasks. The most productive classroom configuration might be pairs, so that students can easily shift their attention back and forth from whole group discussion to their own work.

Launch (Whole Class): Begin the task by introducing the equation: $\log_2(x+b) = \log_2 x + \log_2 b$. Ask students why this might make sense. Expect to hear that they have "distributed" the log.

Without judging the merits of this idea, ask students how they could test the claim. When the idea to test particular numbers comes up, set students to work on questions #1 and #2. After students have had a chance to work on #2, ask a student that has an example showing the statement to be untrue to share his/her work. You may need to help rewrite the student's work so that the statements are clear because this is a strategy that students will want to use throughout the task.

Explore (Small Group): Ask students to turn their attention to question #3. Based on their work with log graphs in previous tasks, they have not seen the graph of a log function with a multiplier in the argument, like $f(x) = \log_2(4x)$. Ask students how the 4 might affect the graph. Because Abe thinks the function is a vertical shift of $y = \log_2 x$, ask student what the equation would look like with a vertical shift so that they generate the idea that the vertical shift typically looks like $y = a + \log_2 x$. Ask students to investigate questions #3 and #4. As students are working, look for a student that has redrawn the x-axis or use a straight edge to show the translation of the graph.

Discuss (Whole Class): When students have had enough to time to find the vertical shift, have a student demonstrate how they were able to tell that the function is a vertical shift up 2. It will help move the class forward in the next part of the task if a student demonstrates redrawing the x-axis so it is easy to see that the graph is a translation, but not a dilation of $y = \log_2 x$. After this discussion, have students work on questions #5, 6, and 7.

Explore (Small Group): While students are working, listen for students who are able to describe the pattern Mary has noticed in the task. Encourage students to test Mary's conjecture with some numbers, just as they did in the beginning of the task.

Discuss (Whole Class): Ask several students to state Mary's rule in their own words. Try to combine the student statements into something like: "The log of a product is the sum of the logs" or give them this statement and ask them to discuss how it describes the pattern they have noticed. Have several students show examples that provide evidence that the statement is true, but remind students that a few examples don't count as a proof. After this discussion, ask students to complete the rest of the task.

Explore (Small Group): Support students as they work to recognize the patterns and express a rule in #9 as both an equation and in words. Students may have difficulty with the notation, so ask them to state the rule in words first, and then help them to write it symbolically.

Discuss (Whole Class): The remaining discussion should follow each of the questions in the task from #9 onward. As the discussion progresses, show student examples of each rule, both to provide evidence that the rule is true and also to practice using the rule. Emphasize reasoning that helps students to see that the log rules are like the exponent rules because of the relationship between logs and exponents.

Aligned Ready, Set, Go: ***Logarithmic Functions 2.3***

READY, SET, GO! Name Period Date

READY

Topic: Recalling fractional exponents

Write the following with an exponent. Simplify when possible.

1. $\sqrt[5]{x}$
2. $\sqrt[7]{s^2}$
3. $\sqrt[3]{w^8}$
4. $\sqrt[3]{8r^6}$
5. $\sqrt[5]{125m^5}$
6. $\sqrt[3]{(8x)^2}$
7. $\sqrt[3]{9b^8}$
8. $\sqrt{75x^6}$

Rewrite with a fractional exponent. Then find the answer.

9. $log_3\sqrt[5]{3} =$
10. $log_2\sqrt[3]{4} =$
11. $log_7\sqrt[5]{343} =$
12. $log_5\sqrt[5]{3125} =$

SET

Topic: Using the properties of logarithms to expand logarithmic expressions

Use the properties of logarithms to expand the expression as a sum or difference, and/or constant multiple of logarithms. (Assume all variables are positive.)

13. $log_5 7x$
14. $log_5 10a$
15. $log_5 \frac{5}{b}$
16. $log_5 \frac{d}{4}$
17. $log_6 x^3$
18. $log_5 9x^2$
19. $log_2 (7x)^4$
20. $log_3 \sqrt{w}$

21. $log_5 \frac{xyz}{w}$

22. $log_5 \frac{9\sqrt{x}}{y^3}$

23. $log_2 \left(\frac{x^2-4}{x^3}\right)$

24. $log_2 \left(\frac{x^2}{y^5 w^3}\right)$

GO

Topic: Writing expressions in exponential form and logarithmic form

Convert to logarithmic form.

25. $2^9 = 512$

26. $10^{-2} = 0.01$

27. $\left(\frac{2}{3}\right)^{-1} = \frac{3}{2}$

Write in exponential form.

28. $log_4 2 = \frac{1}{2}$

29. $log_{\frac{1}{3}} 3 = -1$

30. $log_{\frac{2}{5}} \frac{8}{125} = 3$

2.4 Log-Arithm-etic

A Practice Understanding Task

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Abe and Mary are feeling good about their *log* rules and bragging about their mathematical prowess to all their friends when this exchange occurs:

Stephen: I guess you think you're pretty smart because you figured out some *log* rules, but I want to know what they're good for.

Abe: Well, we've seen a lot of times when equivalent expressions are handy. Sometimes when you write an expression with a variable in it in a different way it means something different.

1. What are some examples from your previous experience where equivalent expressions were useful?

Mary: I was thinking about the *Log Logic* task where we were trying to estimate and order certain *log* values. I was wondering if we could use our log rules to take values we know and use them to find values that we don't know.

For instance: Let's say you want to calculate $\log_2 6$. If you know what $\log_2 2$ and $\log_2 3$ are then you can use the product rule and say:

$$\log_2(2 \cdot 3) = \log_2 2 + \log_2 3$$

Stephen: That's great. Everyone knows that $\log_2 2 = 1$, but what is $\log_2 3$?

Abe: Oh, I saw this somewhere. Uh, $\log_2 3 = 1.585$. So Mary's idea really works. You say:

$$\log_2(2 \cdot 3) = \log_2 2 + \log_2 3$$

$$= 1 + 1.585$$

$$= 2.585$$

$$\log_2 6 = 2.585$$

2. Based on what you know about logarithms, explain why 2.585 is a reasonable value for $\log_2 6$?

Stephen: Oh, oh! I've got one. I can figure out $\log_2 5$ like this:

$$\log_2(2+3) = \log_2 2 + \log_2 3$$

$$= 1 + 1.585$$

$$= 2.585$$

$$\log_2 5 = 2.585$$

3. Can Stephen and Mary both be correct? Explain who's right, who's wrong (if anyone) and why.

Now you can try applying the *log* rules yourself. Use the values that are given and the ones that you know by definition, like $\log_2 2 = 1$, to figure out each of the following values. Explain what you did in the space below each question.

$\log_2 3 = 1.585$ $\log_2 5 = 2.322$ $\log_2 7 = 2.807$

The three rules, written for any base $b > 1$ are:

Log of a Product Rule: $\log_b(xy) = \log_b x + \log_b y$

Log of a Quotient Rule: $\log_b\left(\frac{x}{y}\right) = \log_b x - \log_b y$

Log of a Power Rule: $\log_b(x^k) = k \log_b x$

4. $\log_2 9 =$ ______________________

5. $\log_2 10 =$ ______________________

6. $\log_2 12 =$ ______________________

7. $\log_2\left(\frac{7}{3}\right) =$ ______________________________

8. $\log_2\left(\frac{30}{7}\right) =$ ______________________________

9. $\log_2 486 =$ ______________________________

10. Given the work you have just done, what other values would you need to figure out the value of the base 2 log for any number?

Sometimes thinking about equivalent expressions with logarithms can get tricky. Consider each of the following expressions and decide if they are always true for the numbers in the domain of the logarithmic function, sometimes true, or never true. Explain your answers. If you answer "sometimes true", describe the conditions that must be in place to make the statement true.

11. $\log_4 5 - \log_4 x = \log_4\left(\frac{5}{x}\right)$ ____________________________________

12. $\log 3 - \log x - \log x = \log\left(\frac{3}{x^2}\right)$ ____________________________________

13. $\log x - \log 5 = \frac{\log x}{\log 5}$ ____________________________________

14. $5\log x = \log x^5$ ____________________________________

15. $2\log x + \log 5 = \log(x^2 + 5)$ ____________________________________

16. $\frac{1}{2}\log x = \log\sqrt{x}$ ____________________________________

17. $\log(x - 5) = \frac{\log x}{\log 5}$ ____________________________________

2.4 Log-Arithm-etic – Teacher Notes

A Practice Understanding Task

Purpose:
The purpose of this task is to extend student understanding of log properties and using the properties to write equivalent expressions. In the beginning of the task, students are given values of a few log expressions and asked to use log properties and known values of log expressions to find unknown values. This is an opportunity to see how the known log values can be used and to practice using logarithms and substitution. In the second part of the task, students are asked to determine if the given equations are always true (in the domain of the expression), sometimes true, or never true. This gives students an opportunity to work through some common misconceptions about log properties and to write equivalent expressions using logs.

Core Standards Focus:

F.IF.8. Write a function defined by an expression in different but equivalent forms to reveal and explain different properties of the function.

F.LE.4. For exponential models, express as a logarithm the solution to $ab^{ct} = d$ where a, c, and d are numbers and the base b is 2, 10, or e; evaluate the logarithm using technology.

Note to F.LE.4: *Consider extending this unit to include the relationship between properties of logarithms and properties of exponents, such as the connection between the properties of exponents and the basic logarithm property that log(xy) = log x + log y.*

Standards for Mathematical Practice:

SMP 2 – Reason abstractly and quantitatively

SMP 6 – Attend to precision

The Teaching Cycle

Launch (Whole Class): Launch the task by reading through the scenario and asking students to work problem #1. Follow it by a discussion of their answers, pointing out that equivalent forms often have different meanings in a story context and they can be helpful in solving equations and graphing. Follow this short discussion by having students work problems #2 and #3, then discussing them as a class. The purpose of question #2 is to demonstrate how to use the log rules to find values, and emphasize how they can use the definition of a logarithm to determine if the value they find is reasonable. After discussing these two problems, students should be ready to use the properties to find values of logs. Have students work questions #4-10 before coming back for a discussion.

Explore (Small Group): As students are working, they may need support in finding combinations of factors to use so they can apply the log properties. You may want to remind them of using factor trees or a similar strategy for breaking down a number into its factors. Watch for two students that use a different combination of factors to find the value they are looking for. As you are monitoring student work, be sure they are using good notation to communicate how they are finding the values.

Discuss (Whole Class): Discuss a few of the problems, selecting those that caused controversy among students as they worked. For each problem, be sure to demonstrate the way to use notation and the log properties to find the values. An example might be:

$$\log_2\left(\frac{30}{7}\right) = \log_2 30 - \log_2 7$$

$$= \log_2(5 \cdot 2 \cdot 3) - \log_2 7$$

$$= \log_2 5 + \log_2 2 + \log_2 3 - \log_2 7$$

$$= 2.322 + 1 + 1.585 - 2.807$$

$$= 2.1$$

After finding each value, discuss whether or not the answer is reasonable. After a few of these problems, turn students' attention to the remainder of the task.

Explore (Small Group): Support students as they work in making sense of the statements and verifying them. The statements are designed to bring out misconceptions, so discussion among students should be encouraged. There are several possible strategies for verifying these equations, including using the log properties to manipulate one side of the equation to match the other or trying to put in numbers to the statement. Look for both types of strategies so that the numerical approach can provide evidence, but the algebraic approach can prove (or disprove) the statement.

Discuss (Whole Group): Again, select problems for discussion that have generated controversy or exposed misconceptions. It will often be useful to test the statement with numbers, although that may be difficult for students in some cases. Encourage students to cite the log property they are using as they manipulate the statements to show equivalence. For statements that are never true, ask students how they might correct the statement to make it true.

11. Always true
12. Always true
13. Never true
14. Always true
15. Never true
16. Always true
17. Never true

Aligned Ready, Set, Go: ***Logarithmic Functions 2.4***

READY, SET, GO! Name Period Date

READY

Topic: Solving simple exponential and logarithmic equations

You have solved exponential equations before based on the idea that $\boldsymbol{a^x = a^y, if\ and\ only\ if\ x = y}$.

You can use the same logic on logarithmic equations. $\boldsymbol{log_a x = log_a y, if\ and\ only\ if\ x = y}$

Rewrite each equation to set up a one-to-one correspondence between all of the parts. Then solve for *x*.

Example: ***Original equation***	***Rewritten equation:***	***Solution:***
a.) $3^x = 81$	$3^x = 3^4$	$x = 4$
b.) $log_2 x - log_2 5 = 0$	$log_2 x = log_2 5$	$x = 5$

1. $3^{x+4} = 243$

2. $\left(\frac{1}{2}\right)^x = 8$

3. $\left(\frac{3}{4}\right)^x = \frac{27}{64}$

4. $log_2 x - log_2 13 = 0$

5. $log_2(2x - 4) - log_2 8 = 0$

6. $log_2(x + 2) - log_2 9x = 0$

7. $\frac{log\ 2x}{log\ 14} = 1$

8. $\frac{log(5x-1)}{log\ 29} = 1$

9. $\frac{log\ 5^{(x-2)}}{log\ 625} = 1$

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SET

Topic: Rewriting logs in terms of known logs

Use properties of logarithms to rewrite the indicated logarithms in terms of the given values, then find the indicated logarithm with using a calculator.

Do not use a calculator to evaluate the logarithms.

Given: $\boldsymbol{log\,16 \approx 1.2}$
$\boldsymbol{log\,5 \approx 0.7}$
$\boldsymbol{log\,8 \approx 0.9}$

10. Find $log\,\frac{5}{8}$

11. Find $log\,25$

12. Find $log\,\frac{1}{2}$

13. Find $log\,80$

14. Find $log\,\frac{1}{64}$

Given $\boldsymbol{log_3 2 \approx 0.6}$
$\boldsymbol{log_3 5 \approx 1.5}$

15. Find $log_3 16$

16. Find $log_3 108$

17. Find $log_3 \frac{3}{50}$

18. Find $log_3 \frac{8}{15}$

19. Find $log_3 486$

20. Find $log_3 18$

21. Find $log_3 120$

22. Find $log_3 \frac{32}{45}$

GO

Topic: Using the definition of logarithm to solve for x

Use your calculator and the definition of *log x* (recall the base is 10) to find the value of *x*.
(Round your answers to 4 decimals.)

23. $log\, x = -3$

24. $log\, x = 1$

25. $log\, x = 0$

26. $log\, x = \frac{1}{2}$

27. $log\, x = 1.75$

28. $log\, x = -2.2$

29. $log\, x = 3.67$

30. $log\, x = \frac{3}{4}$

31. $log\, x = 6$

2.5 Powerful Tens

A Practice Understanding Task

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Table Puzzles

1. Find the missing values of x in the tables:

a.

x	$y = 10^x$
-2	$\frac{1}{100}$
1	10
	50
	100
3	1000

b.

x	$y = 3(10^x)$
	0.3
0	3
	94.87
2	300
	1503.56

c. What equations could be written, in terms of x only, for each of the rows that are missing the x in the two tables above?

d.

x	$y = \log x$
0.01	-2
	-1
10	1
	1.6
100	2

e.

x	$y = \log(x+3)$
	-2
-2.9	-1
	0.3
7	1
	3

f. What equations could be written, in terms of x only, for each of the rows that are missing the x in the two tables above?

2. What strategy did you use to find the solutions to equations generated by the tables that contained exponential functions?

3. What strategy did you use to find the solutions to equations generated by the tables that contained logarithmic functions?

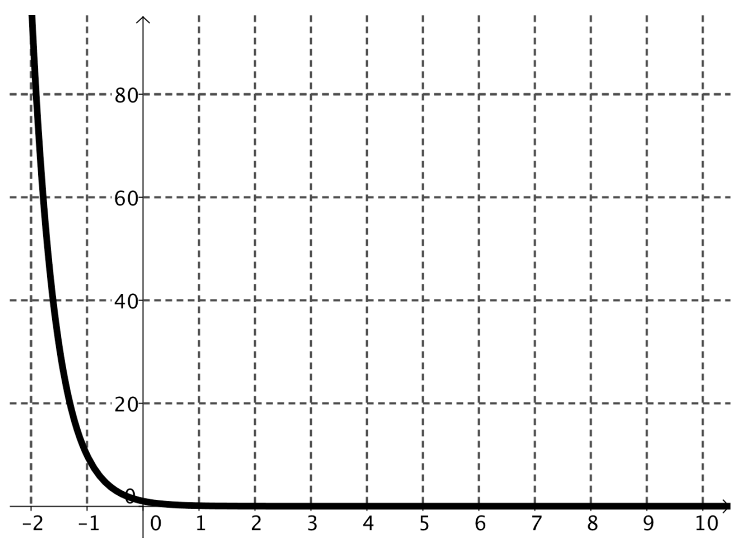
Graph Puzzles

4. The graph of $y = 10^{-x}$ is given below. Use the graph to solve the equations for x and label the solutions.

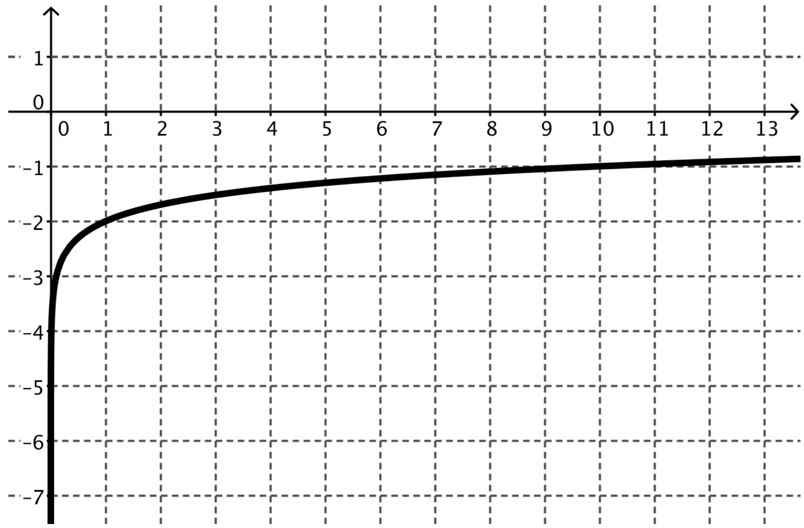
a. $40 = 10^{-x}$
$x =$ ____
Label the solution with an A on the graph.

b. $10^{-x} = 10$
$x =$ ____
Label the solution with a B on the graph.

c. $10^{-x} = 0.1$
$x =$ ____
Label the solution with a C on the graph.



5. The graph of $y = -2 + \log x$ is given below. Use the graph to solve the equations for x and label the solutions.



a. $-2 + \log x = -2$

$x =$ ____

Label the solution with an A on the graph.

b. $-2 + \log x = 0$

$x =$ ____

Label the solution with a B on the graph.

c. $-4 = -2 + \log x$

$x =$ ____

Label the solution with a C on the graph.

d. $-1.3 = -2 + \log x$

$x =$ ____

Label the solution with a D on the graph.

e. $1 = -2 + \log x$

$x =$ ____

6. Are the solutions that you found in #5 exact or approximate? Why?

Equation Puzzles

Solve each equation for x

7. 10^x=10,000

8. $125 = 10^x$

9. $10^{x+2} = 347$

10. $5(10^{x+2}) = 0.25$

11. $10^{-x-1} = \frac{1}{36}$

12. $-(10^{x+2}) = 16$

2.5 Powerful Tens – Teacher Notes

A Practice Understanding Task

Note: Calculators or other technology with base 10 logarithmic and exponential functions are required for this task.

Purpose:

The purpose of this task is to develop student ideas about solving exponential equations that require the use of logarithms and solving logarithmic equations. The task begins with students finding unknown values in tables and writing corresponding equations. In the second part of the task, students use graphs to find equation solutions. Finally, students build on their thinking with tables and graphs to solve equations algebraically. All of the logarithmic and exponential equations are in base 10 so that students can use technology to find values.

Core Standards Focus:

F.LE.4. For exponential models, express as a logarithm the solution to $ab^{ct} = d$ where a, c, and d are numbers and the base b is 2, 10, or e; evaluate the logarithm using technology.

Standards for Mathematical Practice:

SMP 5 – Use appropriate tools strategically

SMP 6 – Attend to precision

The Teaching Cycle:

Launch (Whole Class):

Remind students that they are very familiar with constructing tables for various functions. In previous tasks, they have selected values for *x* and calculated the value of *y* based upon an equation or other representation. They have also constructed graphs based upon having an equation or a set of *x* and *y* values. In this task they will be using tables and graphs to work in reverse, finding the *x* value for a given *y*.

Explore (Small Group):

Monitor students as they work and listen to their strategies for finding the missing values of x. As they are working on the table puzzles, encourage them to consider writing equations as a way to track their strategies. In the graph puzzles, they will find they can only get approximate answers on a few equations. Encourage them to use the graph to estimate a value and to interpret the solution in the equation. The purpose of the tables and graphs is to help students draw upon their thinking from previous tasks to solve the equations. Remind students to connect the ideas as they work on the equation puzzles.

Discuss (Whole Class):

Start the discussion with a student that has written and solved an equation for the third row in table b. The equation written should be:

$$\mathbf{94.87 = 3(10^{x})}$$

Ask the student to describe how they wrote the equation and then their strategy for solving it. Be sure to have students describe their thinking about how to unwind the function as the steps are tracked on the equation. Ask the class where this point would be on a graph of the function. Ask students what the graph of the function would look like, and they should be able to describe a base 10 exponential function with a vertical stretch of 3.

Move the discussion to table "e", focusing on the last row of the table. Again, have students write the equation:

$$\mathbf{3 = \log(x + 3)}$$

Ask the presenting student to describe his/her thinking about how to find the value of x in the table and, once again, track the steps algebraically. There are a couple of likely mistakes made by students who have tried to solve this equation algebraically. If they arise during your observation of students, discuss them here. Again, connect the solution to the graph of the function. Students should be noticing that since logs and exponentials are inverse functions, exponential equations can be solved with logs and log equations are solved with exponentials.

Move the discussion to the graph of $y = 10^{-x}$. Ask students to describe how they used the graph to find the solution to "a". Ask students how they could check the solution in the equation. Does the solution they found with the graph make sense? How would they solve this equation without a graph? Track the steps algebraically, showing something like the following:

$$40 = 10^{-x}$$

$$\log 40 = \log(10^{-x})$$

$$1.602 = -x$$

(Make sure students can explain this step, both using the calculator and simplifying the right side of the equation. It would be useful if students noticed they could use the log properties to rewrite the right side of the equation as $-x(\log 10)$ in addition to using the definition of the logarithm.)

$$x = -1.602$$

Finally, ask students to show solutions to as many of the equation puzzles that time will allow. In every case, be sure students can describe how they use logs to undo the exponential and that their notation matches their thinking.

Aligned Ready, Set, Go: ***Logarithmic Functions 2.5***

READY, SET, GO! Name Period Date

READY

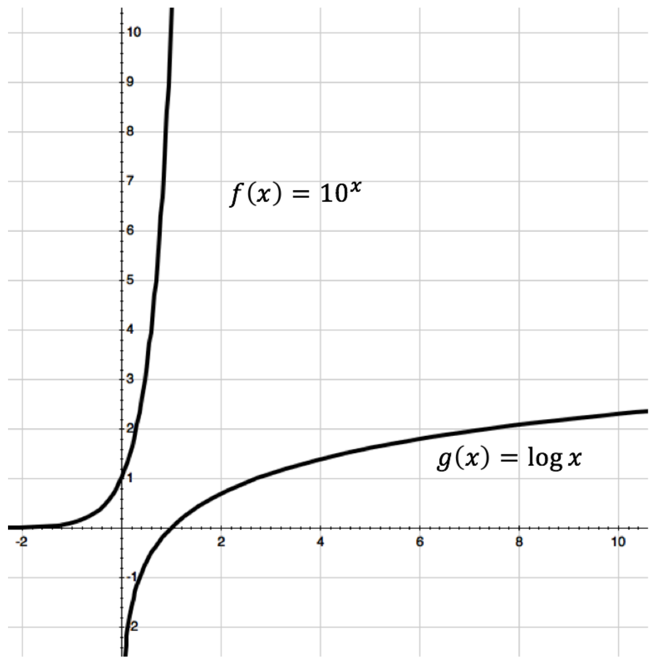
Topic: Comparing the graphs of the exponential and logarithmic functions

The graphs of $f(x) = 10^x$ and $g(x) = log\, x$ are shown in the same coordinate plane.

Make a list of the characteristics of each function.

1. $f(x) = 10^x$

2. $g(x) = log\, x$



Each question below refers to the graphs of the functions $f(x) = 10^x$ and $g(x) = log\, x$. State whether they are true or false. If they are false, correct the statement so that it is true.

_________ 3. Every graph of the form $g(x) = log\, x$ will contain the point $(1, 0)$.

_________ 4. Both graphs have vertical asymptotes.

_________ 5. The graphs of $f(x)$ and $g(x)$ have the same rate of change.

_________ 6. The functions are inverses of each other.

_________ 7. The range of $f(x)$ is the domain of $g(x)$.

_________ 8. The graph of $g(x)$will never reach 3.

SET

Topic: Solving logarithmic equations *(base 10)* by taking the log of each side

Evaluate the following logarithms

9. $log\,10$
10. $log\,10^{-7}$
11. $log\,10^{75}$
12. $log\,10^{x}$
13. $log_3 3^5$
14. $log_8 8^{-3}$
15. $log_{11} 11^{37}$
16. $log_m m^n$

You can use this property of logarithms to help you solve logarithmic equations.
**Note: This property only works when the base of the logarithm matches the base of the exponent.*

Solve the equations by inserting log on both sides of the equation. (You will need a calculator.)

17. $10^n = 4.305$
18. $10^n = 0.316$
19. $10^n = 14{,}521$
20. $10^n = 483.059$

GO

Topic: Solving equations involving compound interest

Formula for compound interest: If *P* dollars is deposited in an account paying an annual rate of interest *r* compounded (paid) *n* times per year, the account will contain $\boldsymbol{A = P\left(1+\frac{r}{n}\right)^{nt}}$ dollars after *t* years.

21. How much money will there be in an account at the end of 10 years if $3000 is deposited at 6% annual interest compounded as follows: (Assume no withdrawals are made.)
    a.) annually
    b.) semiannually
    c.) quarterly
    d.) daily (Use *n* = 365.)

22. Find the amount of money in an account after 12 years if $5,000 is deposited at 7.5% annual interest compounded as follows: (Assume no withdrawals are made.)
    a.) annually
    b.) semiannually
    c.) quarterly
    d.) daily (Use *n* = 365.)