

*Transforming Mathematics Education*

# ALGEBRA II

*An Integrated Approach*

Standard Teacher Notes

MODULE 2 HONORS

# Logarithmic Functions

MATHEMATICSVISIONPROJECT.ORG

**The Mathematics Vision Project**

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## 2.1 Log Logic

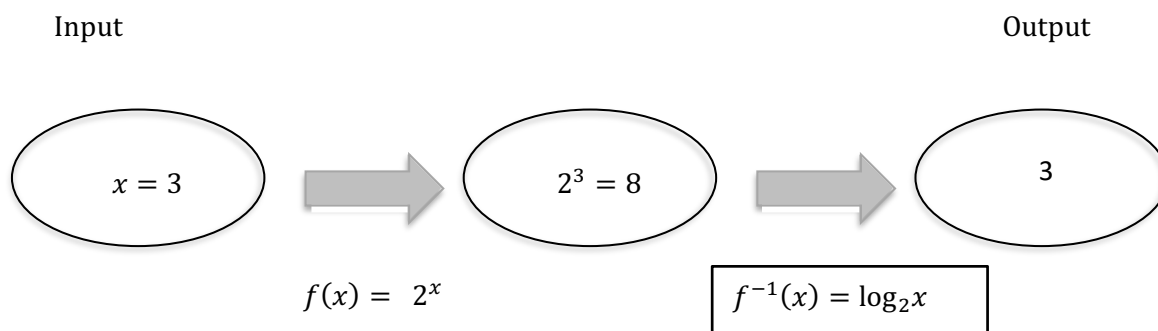
### A Develop Understanding Task

We began thinking about logarithms as inverse functions for exponentials in *Tracking the Tortoise*. Logarithmic functions are interesting and useful on their own. In the next few tasks, we will be working on understanding logarithmic expressions, logarithmic functions, and logarithmic operations on equations.



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We showed the inverse relationship between exponential and logarithmic functions using a diagram like the one below:



We could summarize this relationship by saying:

$$2^3 = 8 \quad \text{so,} \quad \log_2 8 = 3$$

Logarithms can be defined for any base used for an exponential function. Base 10 is popular. Using base 10, you can write statements like these:

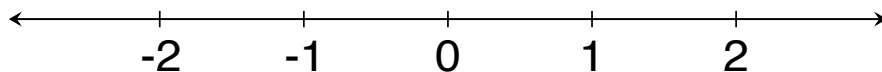
$$\begin{aligned} 10^1 &= 10 & \text{so,} & \log_{10} 10 = 1 \\ 10^2 &= 100 & \text{so,} & \log_{10} 100 = 2 \\ 10^3 &= 1000 & \text{so,} & \log_{10} 1000 = 3 \end{aligned}$$

The notation may see different, but you can see the inverse pattern where the inputs and outputs switch.

The next few problems will give you an opportunity to practice thinking about this pattern and possibly make a few conjectures about other patterns related to logarithms.

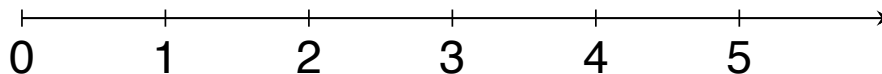
Place the following expressions on the number line. Use the space below the number line to explain how you knew where to place each expression.

1. A.  $\log_3 3$       B.  $\log_3 9$       C.  $\log_3 \left(\frac{1}{3}\right)$       D.  $\log_3 1$       E.  $\log_3 \left(\frac{1}{9}\right)$



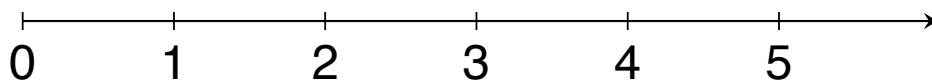
Explain: \_\_\_\_\_

2. A.  $\log_3 81$       B.  $\log_{10} 100$       C.  $\log_8 8$       D.  $\log_5 25$       E.  $\log_2 32$



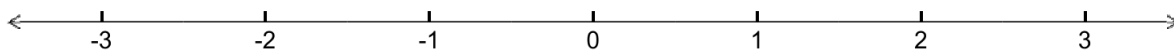
Explain: \_\_\_\_\_

3. A.  $\log_7 7$       B.  $\log_9 9$       C.  $\log_{11} 1$       D.  $\log_{10} 1$



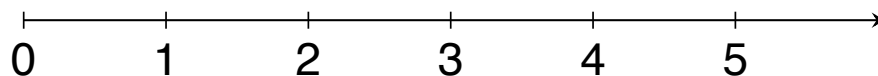
Explain: \_\_\_\_\_

4. A.  $\log_2 \left(\frac{1}{4}\right)$       B.  $\log_{10} \left(\frac{1}{1000}\right)$       C.  $\log_5 \left(\frac{1}{125}\right)$       D.  $\log_6 \left(\frac{1}{6}\right)$



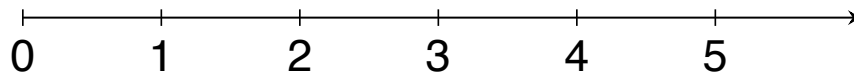
Explain: \_\_\_\_\_

5. A.  $\log_4 16$     B.  $\log_2 16$     C.  $\log_8 16$     D.  $\log_{16} 16$



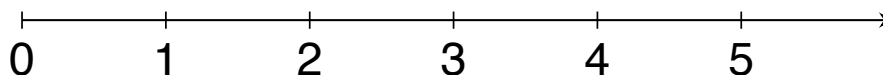
Explain: \_\_\_\_\_

6. A.  $\log_2 5$     B.  $\log_5 10$     C.  $\log_6 1$     D.  $\log_5 5$     E.  $\log_{10} 5$



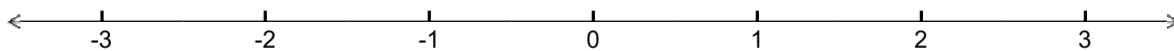
Explain: \_\_\_\_\_

7. A.  $\log_{10} 50$     B.  $\log_{10} 150$     C.  $\log_{10} 1000$     D.  $\log_{10} 500$



Explain: \_\_\_\_\_

8. A.  $\log_3 3^2$     B.  $\log_5 5^{-2}$     C.  $\log_6 6^0$     D.  $\log_4 4^{-1}$     E.  $\log_2 2^3$



Explain: \_\_\_\_\_

Based on your work with logarithmic expressions, determine whether each of these statements is always true, sometimes true, or never true. If the statement is sometimes true, describe the conditions that make it true. Explain your answers.

9. The value of  $\log_b x$  is positive.

Explain: \_\_\_\_\_

10.  $\log_b x$  is not a valid expression if  $x$  is a negative number.

Explain: \_\_\_\_\_

11.  $\log_b 1 = 0$  for any base,  $b > 0$ .

Explain: \_\_\_\_\_

12.  $\log_b b = 1$  for any  $b > 0$ .

Explain: \_\_\_\_\_

13.  $\log_2 x < \log_3 x$  for any value of  $x$ .

Explain: \_\_\_\_\_

14.  $\log_b b^n = n$  for any  $b > 0$ .

Explain: \_\_\_\_\_

## 2.1 Log Logic – Teacher Notes

### *A Develop Understanding Task*

#### **Purpose:**

The purpose of this task is to develop students' understanding of logarithmic expressions and to make sense of the notation. In addition to evaluating log expressions, student will compare expressions that they cannot evaluate explicitly. They will also use patterns they have seen in the task and the definition of a logarithm to justify some properties of logarithms.

#### **Core Standards Focus:**

**F.BF.5 (+)** Understand the inverse relationship between exponents and logarithms and use this relationship to solve problems involving logarithms and exponents.

**F.LE.4** For exponential models, express as a logarithm the solution to  $ab^{ct} = d$  where  $a$ ,  $c$ , and  $d$  are numbers and the base  $b$  is 2, 10, or  $e$ ; evaluate the logarithm using technology.

Note for F.LE.4: *Consider extending this unit to include the relationship between properties of logarithms and properties of exponents, such as the connection between the properties of exponents and the basic logarithm property that  $\log xy = \log x + \log y$ .*

#### **Related Standards: F.BF.4**

#### **Standards for Mathematical Practice:**

**SMP 2 – Reason abstractly and quantitatively**

**SMP 8 – Look for and express regularity in repeated reasoning**

**Vocabulary:** base of a logarithm, argument of a logarithm

#### **The Teaching Cycle:**

#### **Launch (Whole Class):**

Begin by working through each of the examples on page 1 of the task with students. Tell them that since we know that logarithmic functions and exponential functions are inverses, the definition of a logarithm is:

$$\text{If } b^x = n \text{ then } \log_b n = x \text{ for } b > 0$$

Keep this relationship posted where students can refer to it during their work on the task.

**Explore (Small Group):**

The task begins with expressions that will generate integer values. In the beginning, encourage students to use the pattern expressed in the definition to help find the values. If they don't know the powers of the base numbers, they may need to use calculators to identify them. For instance, if they are asked to evaluate  $\log_2 32$ , they may need to use the calculator to find  $2^5 = 32$ . (Author's note: I hope this won't be the case, but the emphasis in this task is on reasoning, not on arithmetic skill.) Thinking about these values will help to review integer exponents.

Starting at #5, there are expressions that can only be estimated and placed on the number line in a reasonable location. Don't give students a way to use the calculator to evaluate these expressions directly; again the emphasis is on reasoning and comparing.

As you monitor students as they work, keep track of students that have interesting justifications for their answers on problems #9 – 15 so that they can be included in the class discussion.

**Discuss (Whole Class):**

Begin the discussion with #2. For each log expression, write the equivalent exponential equation like so:

$$\log_3 81 = 4 \quad 3^4 = 81$$

This will give students practice in seeing the relationship between exponential functions and logarithmic functions. Place each of the values on the number line.

Move the discussion to #4 and proceed in the same way, giving students a brush-up on negative exponents.

Next, work with question #5. Since students can't calculate all of these expressions directly, they will have to use logic to order the expressions. One strategy is to first put the expressions in order from smallest to biggest based on the idea that the bigger the base, the smaller the exponent will need to be to get 16. (Be sure this idea is generalized by the end of the discussion of #5.) Once the numbers are in order, then the approximate values can be considered based upon known values for a particular base.

Work on #7 next. In this problem, the bases are the same, but the arguments are different. The expressions can be ordered based on the idea that for a given base,  $b > 1$ , the greater the argument, the greater the exponent will need to be.

Finally, work through each of problems 9 – 15. This is an opportunity to develop a number of the properties of logarithms from the definitions. After students have justified each of the properties that are always true (#10, 11, 12, and 14), these should be posted in the classroom as agreed-upon properties that can be used in future work.

**Aligned Ready, Set, Go: *Logarithmic Functions 2.1***

READY, SET, GO!

Name \_\_\_\_\_

Period \_\_\_\_\_

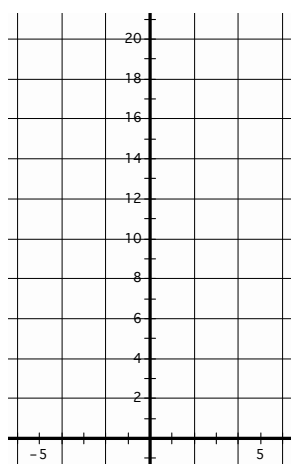
Date \_\_\_\_\_

## READY

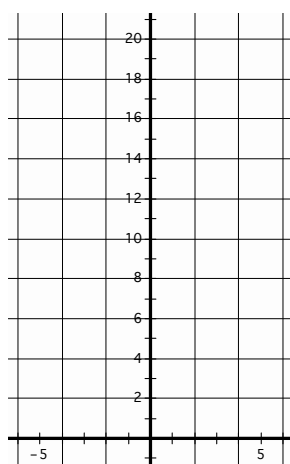
Topic: Graphing exponential equations

Graph each function over the domain  $\{-4 \leq x \leq 4\}$ .

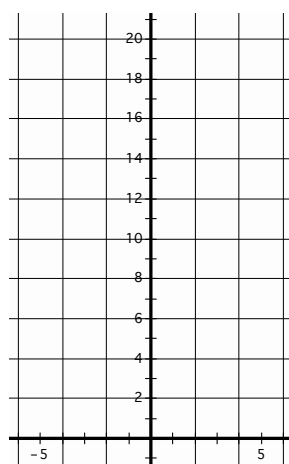
1.  $y = 2^x$



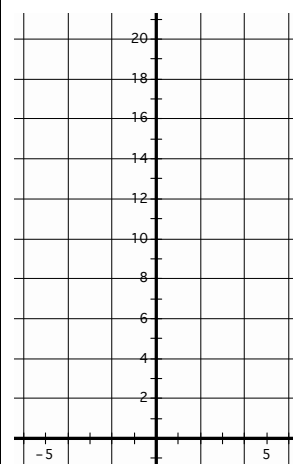
2.  $y = 2 \cdot 2^x$



3.  $y = \left(\frac{1}{2}\right)^x$



4.  $y = 2 \left(\frac{1}{2}\right)^x$



- Compare graph #1 to graph #2. Multiplying by 2 should generate a dilation of the graph, but the graph looks like it has been translated vertically. How do you explain that?
- Compare graph #3 to graph #4. Is your explanation in #5 still valid for these two graphs? Explain.

## SET

Topic: Writing the logarithmic form of an exponential equation.

**Definition of Logarithm:** For all positive numbers  $a$ , where  $a \neq 1$ , and all positive numbers  $x$ ,

$$y = \log_a x \text{ means the same as } x = a^y.$$

(Note the **base** of the exponent and the **base** of the logarithm are both  $a$ .)

Need help? Visit [www.rsgsupport.org](http://www.rsgsupport.org)

7. Why is it important that in the definition of logarithm it is stated that the base of the logarithm does not equal 1?
8. Why is it important that the definition of a logarithm states that the base of the logarithm is positive?
9. Why is it necessary that the definition states that  $x$  in the expression  $\log_a x$  is positive?

Write the following exponential equations in logarithmic form.

| Exponential form                        | Logarithmic form | Exponential form            | Logarithmic form |
|---|------------------|-----------------------------|------------------|
| 10. $5^4 = 625$                         |                  | 11. $3^2 = 9$               |                  |
| 12. $\left(\frac{1}{2}\right)^{-3} = 8$ |                  | 13. $4^{-2} = \frac{1}{16}$ |                  |
| 14. $10^4 = 10000$                      |                  | 15. $a^y = x$               |                  |

16. Compare the exponential form of an equation to the logarithmic form of an equation. What part of the exponential equation is the **answer** to the logarithmic equation?

Topic: Considering values of logarithmic functions

**Answer the following questions. If yes, give an example of the answer. If no, explain why not.**

17. Is it possible for a logarithm to equal a negative number?
18. Is it possible for a logarithm to equal zero?
19. Does  $\log_x 0$  have an answer?
20. Does  $\log_x 1$  have an answer?
21. Does  $\log_x x^5$  have an answer?

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GO

Topic: Reviewing properties of Exponents

**Write each expression as an integer or a simple fraction.**

22.  $27^0$

23.  $11(-6)^0$

24.  $-3^{-2}$

25.  $4^{-3}$

26.  $\frac{9}{2^{-1}}$

27.  $\frac{4^3}{8^0}$

28.  $3\left(\frac{29^3}{11^5}\right)^0$

29.  $\frac{3}{6^{-1}}$

30.  $\frac{32^{-1}}{4^{-1}}$

**Need help?** Visit [www.rsgsupport.org](http://www.rsgsupport.org)

## 2.2 Falling Off a Log

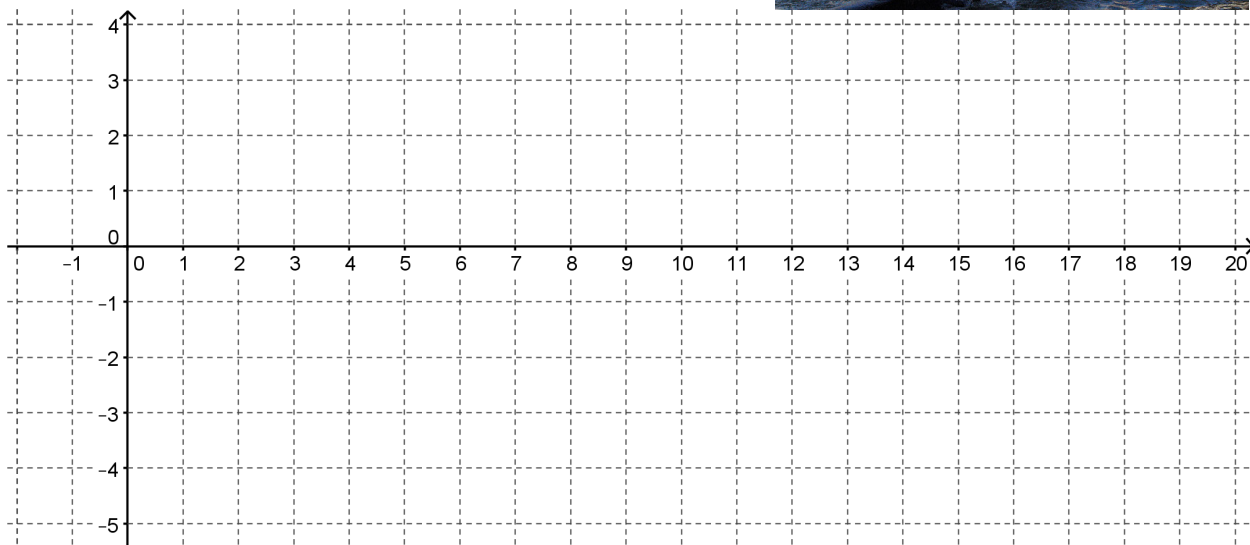
### *A Solidify Understanding Task*

- Construct a table of values and a graph for each of the following functions. Be sure to select at least two values in the interval  $0 < x < 1$ .

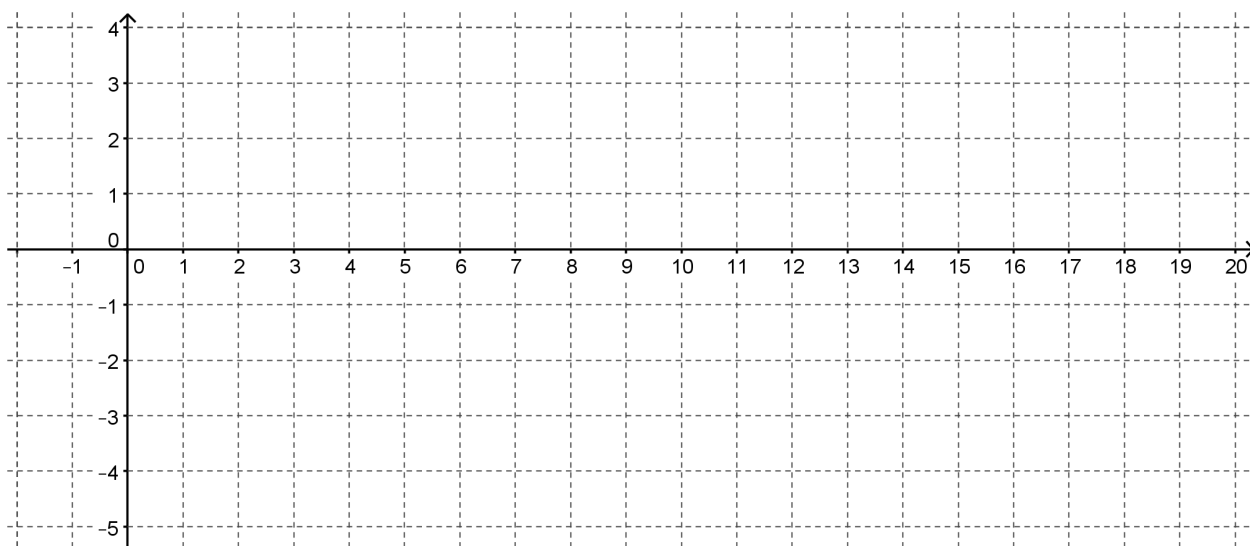
a)  $f(x) = \log_2 x$



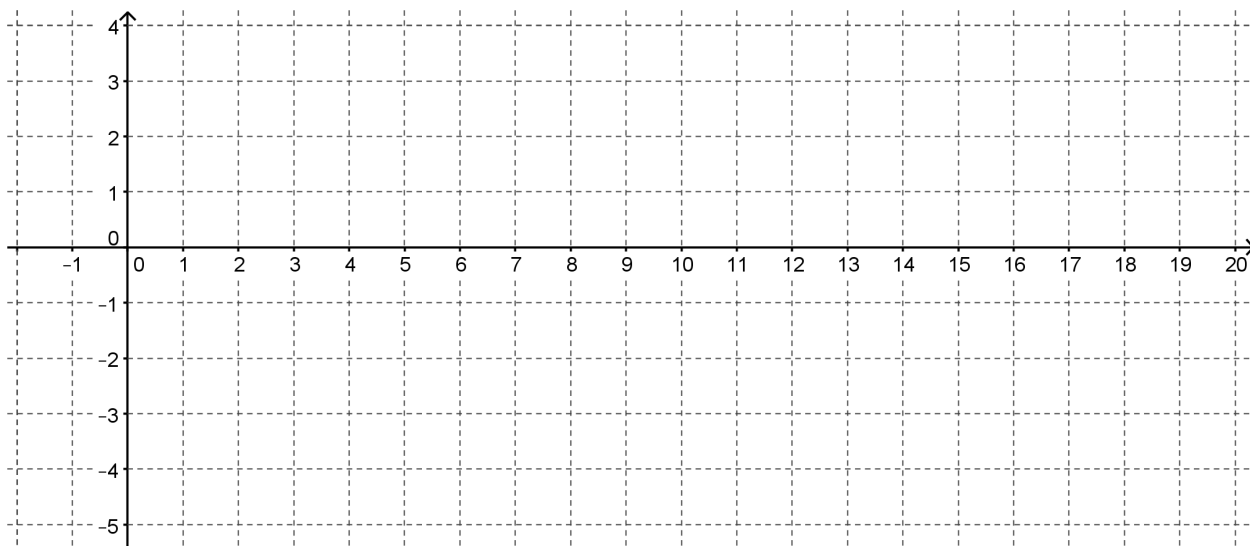
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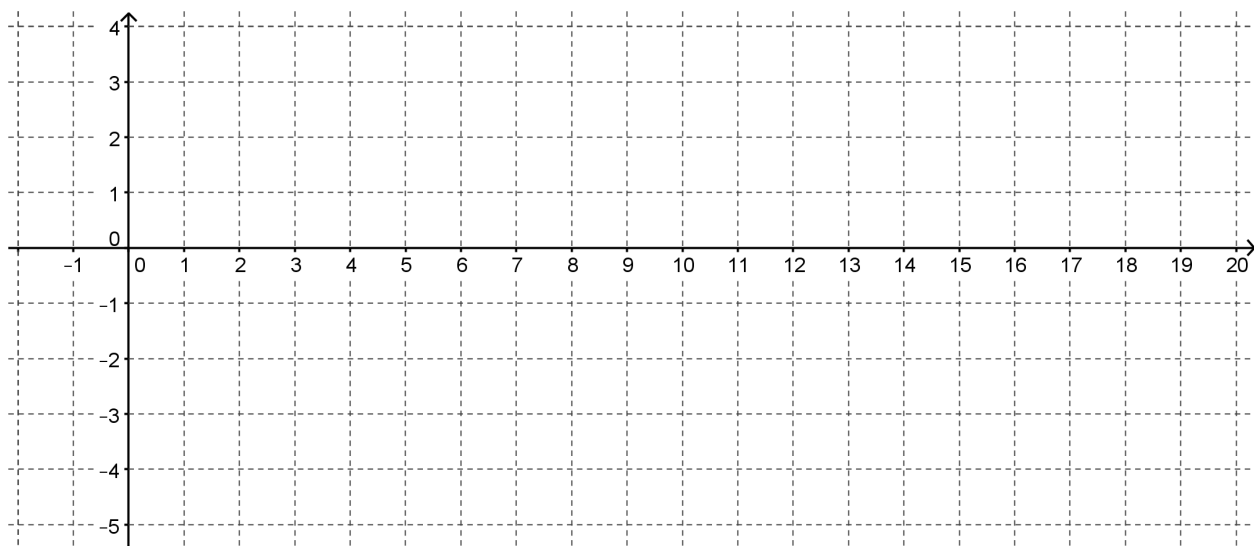
b)  $g(x) = \log_3 x$



c)  $h(x) = \log_4 x$



d)  $k(x) = \log_{10} x$



2. How did you decide what values to use for  $x$  in the table?
3. How did you use the  $x$  values to find the  $y$  values in the table?
4. What similarities do you see in the graphs?

5. What differences do you observe in the graphs?
6. What is the effect of changing the base on the graph of a logarithmic function?

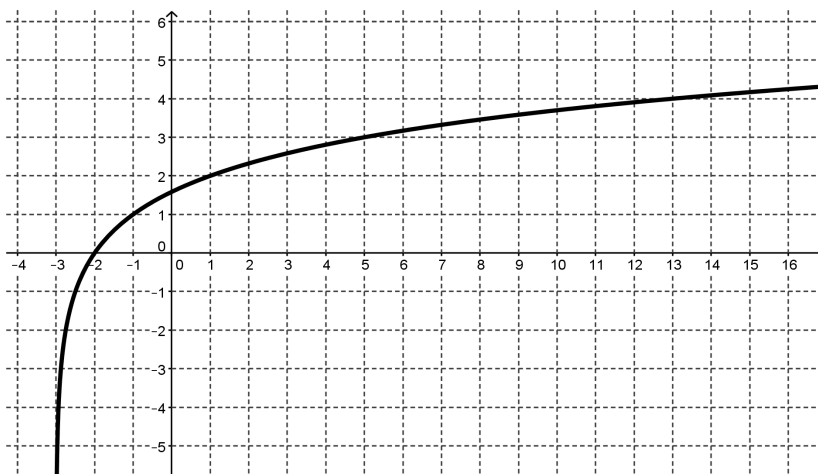
Let's focus now on  $k(x) = \log_{10} x$  so that we can use technology to observe the effects of changing parameters on the function. **Because base 10 is a commonly used base for exponential and logarithmic functions, it is often abbreviated and written without the base, like this:  $k(x) = \log x$ .**

7. Use technology to graph  $y = \log x$ . How does the graph compare to the graph that you constructed?
8. How do you predict that the graph of  $y = a + \log x$  will be different from the graph of  $y = \log x$ ?
9. Test your prediction by graphing  $y = a + \log x$  for various values of  $a$ . What is the effect of  $a$  on the graph? Make a general argument for why this would be true for all logarithmic functions.
10. How do you predict that the graph of  $y = \log(x + b)$  will be different from the graph of  $y = \log x$ ?
11. Test your prediction by graphing  $y = \log(x + b)$  for various values of  $b$ .
  - What is the effect of adding  $b$ ?
  - What will be the effect of subtracting  $b$  (or adding a negative number)?

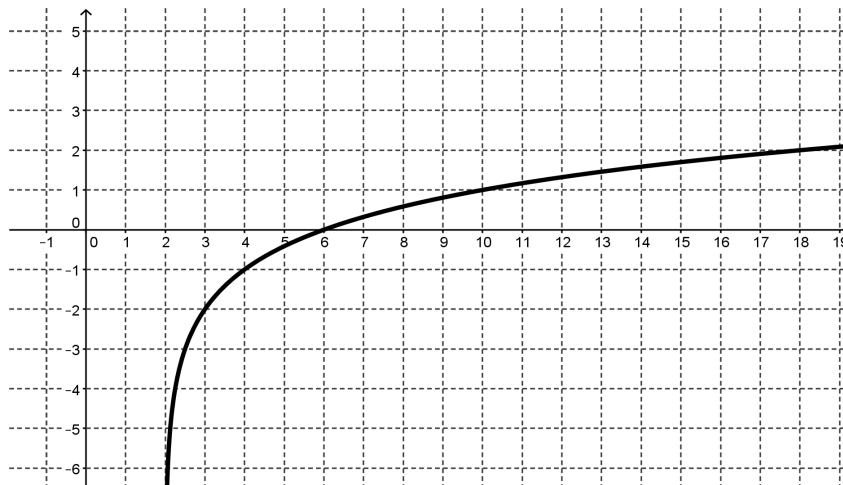
- Make a general argument for why this is true for all logarithmic functions.

12. Write an equation for each of the following functions that are transformations of  $f(x) = \log_2 x$ .

a)

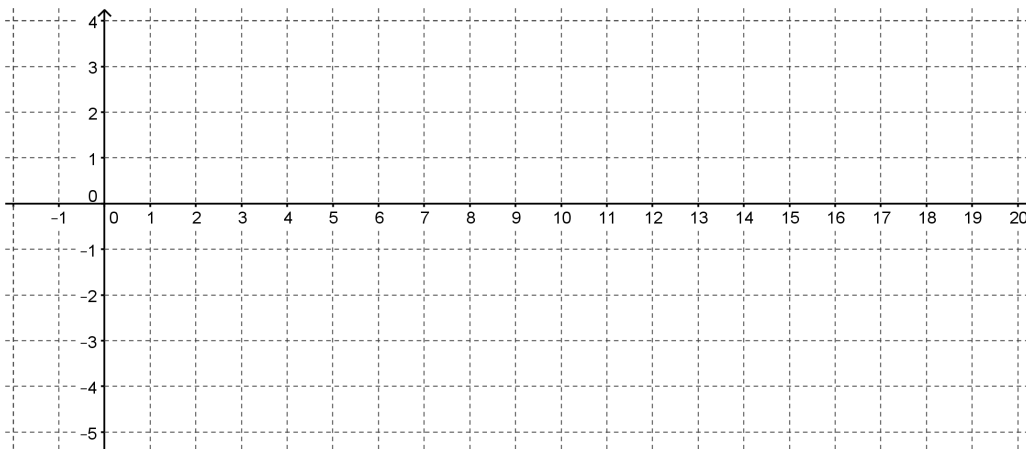


b)

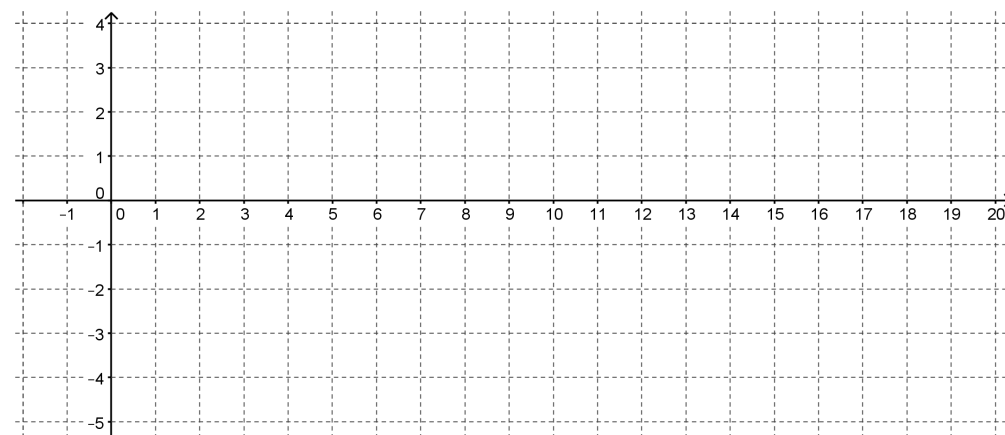


13. Graph and label each of the following functions:

a)  $f(x) = 2 + \log_2(x - 1)$



b)  $g(x) = -1 + \log_2(x + 2)$



14. Compare the transformation of the graphs of logarithmic functions with the transformation of the graphs of quadratic functions.

## 2.2 Falling Off a Log – Teacher Notes

### *A Solidify Understanding Task*

**Purpose:** The purpose of this task is to build on students' understanding of a logarithmic function as the inverse of an exponential function and their previous work in determining values for logarithmic expressions to find the graphs of logarithmic functions of various bases. Students use technology to explore transformations with log graphs in base 10 and then generalize the transformations to other bases.

#### **Core Standards Focus:**

**F.IF.7.a** Graph exponential and logarithmic functions, showing intercepts and end behavior, and trigonometric functions, showing period, midline, and amplitude.

**F.BF.5 (+)** Understand the inverse relationship between exponents and logarithms and use this relationship to solve problems involving logarithms and exponents.

Note for F. BF: Use transformations of functions to find more optimum models as students consider increasingly more complex situations.

For F.BF.3, note the effect of multiple transformations on a single function and the common effect of each transformation across function types. Include functions defined only by a graph.

Extend F.BF.4a to simple rational, simple radical, and simple exponential functions; connect F.BF.4a to F.LE.4.

#### **Related Standards: F.LE.4**

#### **Standards for Mathematical Practice:**

**SMP 2 – Reason abstractly and quantitatively**

**SMP 5 – Use appropriate tools strategically**

**Vocabulary:** vertical asymptote

**Note to Teachers:** Access to graphing technology is necessary for this task.

**A Desmos activity for this task is available at: <https://tinyurl.com/MVPMath3Lesson2-2>. It can be found at Desmos by searching for: *MVP Math III, 2.2 Falling Off a Log*. The activity takes students through questions 7-11, allowing them to use technology to explore the vertical and horizontal shifts on the graphs of a logarithmic function.**

### **The Teaching Cycle**

#### **Launch (Whole Class):**

Begin class by reminding students of the work they did with log expressions in the previous task and soliciting a few exponential and log statements like this:

$$5^3 = 125 \quad \text{so} \quad \log_5 125 = 3$$

Encourage the use of different bases to remind students that the same definition works for all bases,  $b > 1$ . Tell students that in this task they will use what they know about inverses to help them create tables and graph log functions.

#### **Explore (Small Group):**

Monitor students as they work to ensure they are completing both tables and graphs for each function. Some students may choose to graph the exponential function and then reflect it over the  $y = x$  line to get the graph before completing the table. Watch for this strategy and be prepared to highlight it during the discussion. Finding points on the graph for  $0 < x < 1$  may prove difficult for students since negative exponents are often difficult. If students are stuck, remind them that it may be easier to find points on the exponential function and then switch them for the log graphs.

#### **Discuss (Whole Class):**

Begin the discussion with the graph of  $f(x) = \log_2 x$ . Ask a student that used the exponential function  $y = 2^x$  and switched the  $x$  and  $y$  values to present their graph. Then have a student that started by creating a table describe how they obtained the values in the table. Ask the class to identify how the two strategies are connected. By now, students should be able to articulate the idea that powers of 2 are easy values to think about and the value of the log expression will be the exponent in each case.

Move the discussion to question #4, the similarities between the graphs. Students will probably speak generally about the shapes being alike. In the discussions of similarities, be sure that the more technical features of the graphs emerge:

- The point  $(1,0)$  is included
- The domain is  $(0, \infty)$
- The range is  $(-\infty, \infty)$
- The function is increasing over the entire domain.

Ask students to connect each of these features with the definition of a logarithm and properties of inverse functions. At this point, introduce the idea of a vertical asymptote. We know that logarithmic functions are not defined for  $x = 0$  because exponential functions approach, but never actually reach 0. Since logarithmic functions are inverses of exponential functions, we would expect their graphs to be reflections over the  $y = x$  line. Also help students to understand why the  $y$ -values of a logarithmic function become very large negative numbers as  $x$ -values become closer to 0.

Ask what conclusions they could draw about the effect of changing the base on graph. How do these conclusions connect to the strategies they used to order log expressions with different bases in the previous task?

Ask students how the graphs were transformed when a number is added outside the log functions versus inside the argument of the log function. Students should notice that this is just like other functions that they are familiar with such as quadratic functions.

**Aligned Ready, Set, Go: *Logarithmic Functions 2.2***

READY, SET, GO!

Name \_\_\_\_\_

Period \_\_\_\_\_

Date \_\_\_\_\_

**READY**

Topic: Solving simple logarithmic equations

**Find the answer to each logarithmic equation. Then explain how each equation supports the statement, “The answer to a logarithmic equation is always the exponent.”**

1.  $\log_5 625 =$

2.  $\log_3 243 =$

3.  $\log_5 0.2 =$

4.  $\log_9 81 =$

5.  $\log 1,000,000 =$

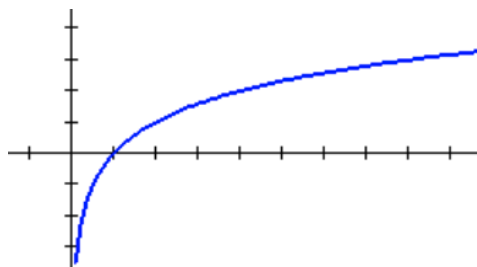
6.  $\log_x x^7 =$

**SET**

Topic: Exploring transformations on logarithmic functions

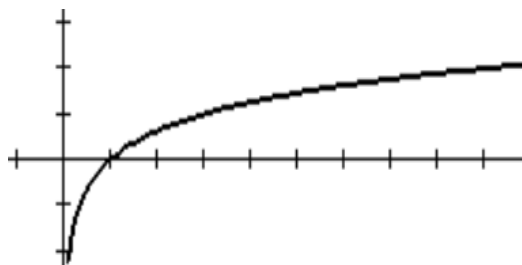
**Answer the questions about each graph.**

7.



- What is the value of  $x$  when  $f(x) = 0$ ?
- What is the value of  $x$  when  $f(x) = 1$ ?
- What is the value of  $f(x)$  when  $x = 2$ ?
- What will be the value of  $x$  when  $f(x) = 4$ ?
- What is the equation of this graph?

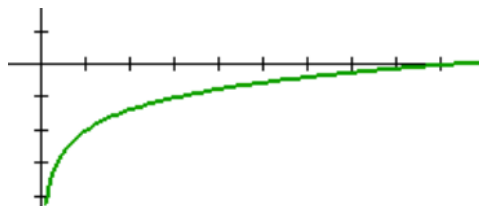
8.



- What is the value of  $x$  when  $f(x) = 0$ ?
- What is the value of  $x$  when  $f(x) = 1$ ?
- What is the value of  $f(x)$  when  $x = 9$ ?
- What will be the value of  $x$  when  $f(x) = 4$ ?
- What is the equation of this graph?

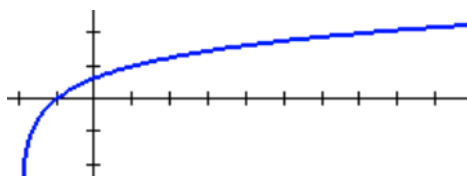
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9. Use the graph and the table of values for the graph to write the equation of the graph. Explain which numbers in the table helped you the most to write the equation.



| X | Y <sub>1</sub> |
|---|----------------|
| 1 | -2             |
| 2 | -1.369         |
| 3 | -1             |
| 5 | -.535          |
| 7 | -.2288         |
| 9 | 0              |

10. Use the graph and the table of values for the graph to write the equation of the graph. Explain which numbers in the table helped you the most to write the equation.



| X  | Y <sub>1</sub> |
|----|----------------|
| -2 | ERROR          |
| -1 | 0              |
| 0  | .63093         |
| 1  | 1              |
| 5  | 1.7712         |
| 6  | 1.8928         |
| 7  | 2              |

### GO

Topic: Using the power to a power rule with exponents

**Simplify each expression. Answers should have only positive exponents.**

11.  $(2^3)^4$
12.  $(x^3)^2$
13.  $(a^3)^{-2}$
14.  $(2^3w)^4$
15.  $(b^{-7})^3$
16.  $(d^{-3})^{-2}$
17.  $x^2 \cdot (x^5)^2$
18.  $m^{-3} \cdot (m^2)^3$
19.  $(x^5)^{-4} \cdot x^{25}$
20.  $(5a^3)^2$
21.  $(6^{-3})^2$
22.  $(2a^3b^2)^2$

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## 2.3 Chopping Logs

### *A Solidify Understanding Task*

Abe and Mary were working on their math homework together when Abe has a brilliant idea!

**Abe:** I was just looking at this *log* function that we graphed in *Falling Off A Log*:

$$y = \log_2(x + b).$$

I started to think that maybe I could just “distribute” the *log* so that I get:

$$y = \log_2 x + \log_2 b.$$

I guess I’m saying that I think these are equivalent expressions, so I could write it this way:

$$\log_2(x + b) = \log_2 x + \log_2 b$$

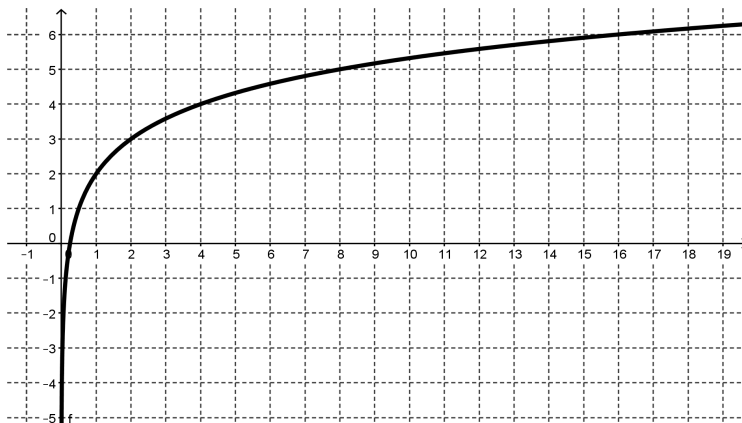
**Mary:** I don’t know about that. *Logs* are tricky and I don’t think that you’re really doing the same thing here as when you distribute a number.

1. What do you think? How can you verify if Abe’s idea works?
2. If Abe’s idea works, give some examples that illustrate why it works. If Abe’s idea doesn’t work, give a counter-example.



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**Abe:** I just know there is something going on with these *logs*. I just graphed  $f(x) = \log_2(4x)$ . Here it is:



It's weird because I think this graph is just a translation of  $y = \log_2 x$ . Is it possible the equation of this graph could be written more than one way?

3. How would you answer Abe's question? Are there conditions that could allow the same graph to have different equations?

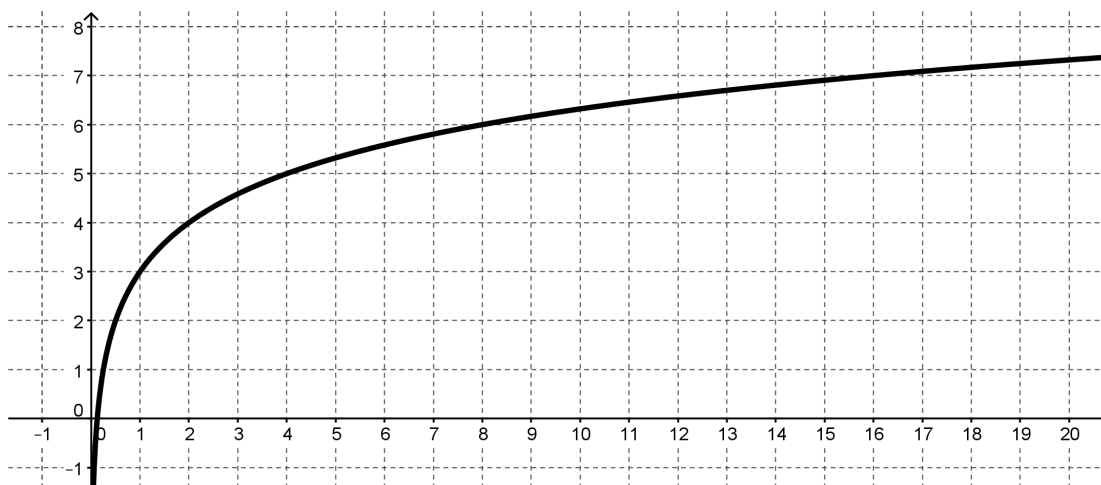
**Mary:** When you say, "a translation of  $y = \log_2 x$ " do you mean that it is just a vertical or horizontal shift? What could that equation be?

4. Find an equation for  $f(x)$  that shows it to be a horizontal or vertical shift of  $y = \log_2 x$ .

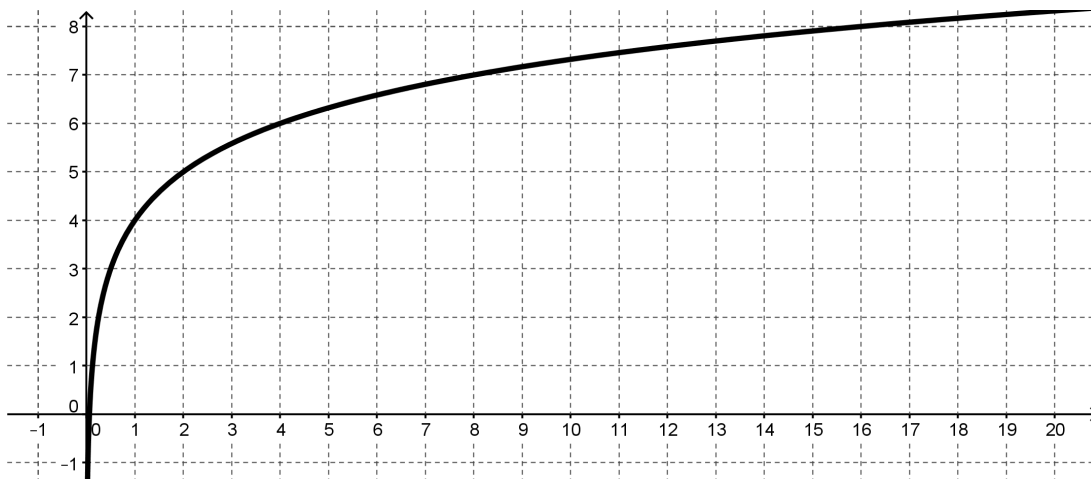
**Mary:** I wonder why the vertical shift turned out to be up 2 when the  $x$  was multiplied by 4. I wonder if it has something to do with the power that the base is raised to, since this is a *log* function. Let's try to see what happens with  $y = \log_2(8x)$  and  $y = \log_2(16x)$ .

5. Try to write an equivalent equation for each of these graphs that is a vertical shift of  $y = \log_2 x$ .

a)  $y = \log_2(8x)$       Equivalent equation: \_\_\_\_\_



b)  $y = \log_2(16x)$       Equivalent equation: \_\_\_\_\_



**Mary:** Oh my gosh! I think I know what is happening here! Here's what we see from the graphs:

$$\log_2(4x) = 2 + \log_2 x$$

$$\log_2(8x) = 3 + \log_2 x$$

$$\log_2(16x) = 4 + \log_2 x$$

Here's the brilliant part: We know that  $\log_2 4 = 2$ ,  $\log_2 8 = 3$ , and  $\log_2 16 = 4$ . So:

$$\log_2(4x) = \log_2 4 + \log_2 x$$

$$\log_2(8x) = \log_2 8 + \log_2 x$$

$$\log_2(16x) = \log_2 16 + \log_2 x$$

I think it looks like the “distributive” thing that you were trying to do, but since you can't really distribute a function, it's really just a *log* multiplication rule. I guess my rule would be:

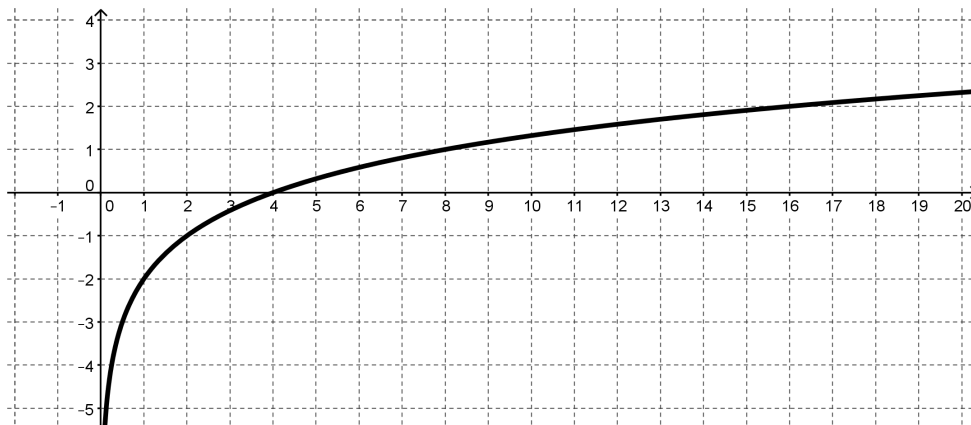
$$\log_2(ab) = \log_2 a + \log_2 b$$

6. How can you express Mary's rule in words?
  
  
  
  
  
  
  
  
  
  
7. Is this statement true? If it is, give some examples that illustrate why it works. If it is not true provide a counter-example.

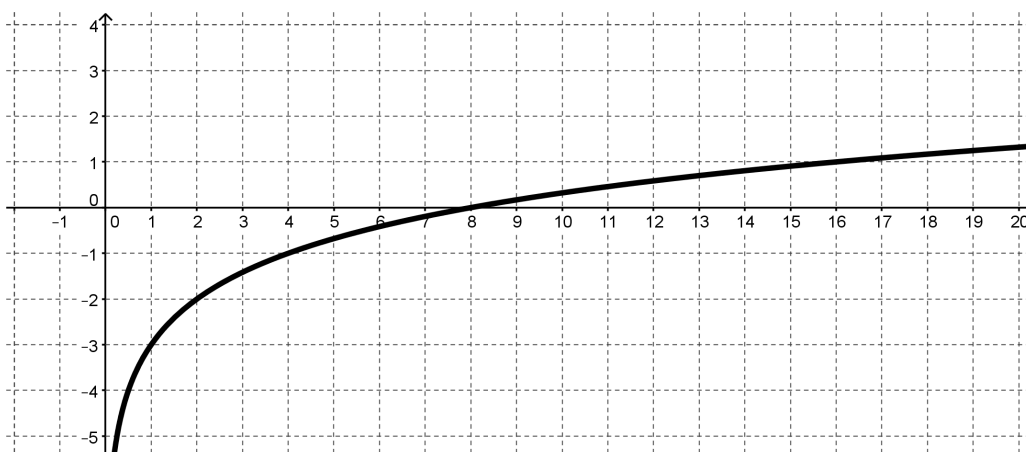
**Mary:** So, I wonder if a similar thing happens if you have division inside the argument of a *log* function. I'm going to try some examples. If my theory works, then all of these graphs will just be vertical shifts of  $y = \log_2 x$ .

8. Here are Abe's examples and their graphs. Test Abe's theory by trying to write an equivalent equation for each of these graphs that is a vertical shift of  $y = \log_2 x$ .

a)  $y = \log_2 \left( \frac{x}{4} \right)$       Equivalent equation: \_\_\_\_\_



b)  $y = \log_2 \left( \frac{x}{8} \right)$       Equivalent equation: \_\_\_\_\_



9. Use these examples to write a rule for division inside the argument of a *log* that is like the rule that Mary wrote for multiplication inside a *log*.

10. Is this statement true? If it is, give some examples that illustrate why it works. If it is not true provide a counter-example.

**Abe:** You're definitely brilliant for thinking of that multiplication rule. But I'm a genius because I've used your multiplication rule to come up with a power rule. Let's say you start with:

$$\log_2(x^3)$$

Really that's the same as having:

$$\log_2(x \cdot x \cdot x)$$

So, I could use your multiplying rule and write:

$$\log_2 x + \log_2 x + \log_2 x$$

I notice there are 3 terms that are all the same. That makes it:

$$3 \log_2 x$$

So my rule is:

$$\log_2(x^3) = 3 \log_2 x$$

If your rule is true, then I have proven my power rule.

**Mary:** I don't think it's really a power rule unless it works for any power. You only showed how it might work for 3.

**Abe:** Oh, good grief! Ok, I'm going to say it can be any number  $x$ , raised to any power,  $k$ . My power rule is:

$$\log_2(x^k) = k \log_2 x$$

Are you satisfied?

11. Provide an argument about Abe's power rule. Is it true or not?

**Abe:** Before we win the Nobel Prize for mathematics I suppose that we need to think about whether or not these rules work for any base.

12. The three rules, written for any base  $b > 1$  are:

**Log of a Product Rule:**  $\log_b(xy) = \log_b x + \log_b y$

**Log of a Quotient Rule:**  $\log_b\left(\frac{x}{y}\right) = \log_b x - \log_b y$

**Log of a Power Rule:**  $\log_b(x^k) = k \log_b x$

Make an argument for why these rules will work in any base  $b > 1$  if they work for base 2.

13. How are these rules similar to the rules for exponents? Why might exponents and *logs* have similar rules?

## 2.3 Chopping Logs – Teacher Notes

### *A Solidify Understanding Task*

**Note to Teachers:** Access to graphing technology is necessary for this task.

**Purpose:** The purpose of this task is to use student understanding of log graphs and log expressions to derive properties of logarithms. In the task students are asked to find equivalent equations for graphs and then to generalize the patterns to establish the product, quotient, and power rules for logarithms.

**Core Standards Focus:**

**F.IF.8.** Write a function defined by an expression in different but equivalent forms to reveal and explain different properties of the function.

**F.LE.4.** For exponential models, express as a logarithm the solution to  $ab^{ct} = d$  where  $a$ ,  $c$ , and  $d$  are numbers and the base  $b$  is 2, 10, or  $e$ ; evaluate the logarithm using technology.

**Note to F.LE.4:** Consider extending this unit to include the relationship between properties of logarithms and properties of exponents, such as the connection between the properties of exponents and the basic logarithm property that  $\log(xy) = \log x + \log y$ .

**Related Standards:** F.BF.5

**Standards for Mathematical Practice:**

**SMP 5 – Use appropriate tools strategically**

**SMP 7 – Look for and make use of structure**

**The Teaching Cycle:**

**Note:** The nature of this task suggests a more guided approach from the teacher than many tasks. The most productive classroom configuration might be pairs, so that students can easily shift their attention back and forth from whole group discussion to their own work.

**Launch (Whole Class):** Begin the task by introducing the equation:  $\log_2(x + b) = \log_2 x + \log_2 b$ . Ask students why this might make sense. Expect to hear that they have “distributed” the log.

Without judging the merits of this idea, ask students how they could test the claim. When the idea to test particular numbers comes up, set students to work on questions #1 and #2. After students have had a chance to work on #2, ask a student that has an example showing the statement to be untrue to share his/her work. You may need to help rewrite the student's work so that the statements are clear because this is a strategy that students will want to use throughout the task.

**Explore (Small Group):** Ask students to turn their attention to question #3. Based on their work with log graphs in previous tasks, they have not seen the graph of a log function with a multiplier in the argument, like  $f(x) = \log_2(4x)$ . Ask students how the 4 might affect the graph. Because Abe thinks the function is a vertical shift of  $y = \log_2 x$ , ask student what the equation would look like with a vertical shift so that they generate the idea that the vertical shift typically looks like  $y = a + \log_2 x$ . Ask students to investigate questions #3 and #4. As students are working, look for a student that has redrawn the x-axis or use a straight edge to show the translation of the graph.

**Discuss (Whole Class):** When students have had enough to time to find the vertical shift, have a student demonstrate how they were able to tell that the function is a vertical shift up 2. It will help move the class forward in the next part of the task if a student demonstrates redrawing the x-axis so it is easy to see that the graph is a translation, but not a dilation of  $y = \log_2 x$ . After this discussion, have students work on questions #5, 6, and 7.

**Explore (Small Group):** While students are working, listen for students who are able to describe the pattern Mary has noticed in the task. Encourage students to test Mary's conjecture with some numbers, just as they did in the beginning of the task.

**Discuss (Whole Class):** Ask several students to state Mary's rule in their own words. Try to combine the student statements into something like: "The log of a product is the sum of the logs" or give them this statement and ask them to discuss how it describes the pattern they have noticed. Have several students show examples that provide evidence that the statement is true, but remind students that a few examples don't count as a proof. After this discussion, ask students to complete the rest of the task.

**Explore (Small Group):** Support students as they work to recognize the patterns and express a rule in #9 as both an equation and in words. Students may have difficulty with the notation, so ask them to state the rule in words first, and then help them to write it symbolically.

**Discuss (Whole Class):** The remaining discussion should follow each of the questions in the task from #9 onward. As the discussion progresses, show student examples of each rule, both to provide evidence that the rule is true and also to practice using the rule. Emphasize reasoning that helps students to see that the log rules are like the exponent rules because of the relationship between logs and exponents.

**Aligned Ready, Set, Go:** *Logarithmic Functions 2.3*

READY, SET, GO!

Name

Period

Date

**READY**

Topic: Recalling fractional exponents

**Write the following with an exponent. Simplify when possible.**

1.  $\sqrt[5]{x}$

2.  $\sqrt[7]{s^2}$

3.  $\sqrt[3]{w^8}$

4.  $\sqrt[3]{8r^6}$

5.  $\sqrt[5]{125m^5}$

6.  $\sqrt[3]{(8x)^2}$

7.  $\sqrt[3]{9b^8}$

8.  $\sqrt{75x^6}$

**Rewrite with a fractional exponent. Then find the answer.**

9.  $\log_3 \sqrt[5]{3} =$

10.  $\log_2 \sqrt[3]{4} =$

11.  $\log_7 \sqrt[5]{343} =$

12.  $\log_5 \sqrt[5]{3125} =$

**SET**

Topic: Using the properties of logarithms to expand logarithmic expressions

**Use the properties of logarithms to expand the expression as a sum or difference, and/or constant multiple of logarithms.** (Assume all variables are positive.)

13.  $\log_5 7x$

14.  $\log_5 10a$

15.  $\log_5 \frac{5}{b}$

16.  $\log_5 \frac{d}{4}$

17.  $\log_6 x^3$

18.  $\log_5 9x^2$

19.  $\log_2 (7x)^4$

20.  $\log_3 \sqrt{w}$

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21.  $\log_5 \frac{xyz}{w}$

22.  $\log_5 \frac{9\sqrt{x}}{y^3}$

23.  $\log_2 \left( \frac{x^2-4}{x^3} \right)$

24.  $\log_2 \left( \frac{x^2}{y^5 w^3} \right)$

**GO**

Topic: Writing expressions in exponential form and logarithmic form

**Convert to logarithmic form.**

25.  $2^9 = 512$

26.  $10^{-2} = 0.01$

27.  $\left( \frac{2}{3} \right)^{-1} = \frac{3}{2}$

**Write in exponential form.**

28.  $\log_4 2 = \frac{1}{2}$

29.  $\log_{\frac{1}{3}} 3 = -1$

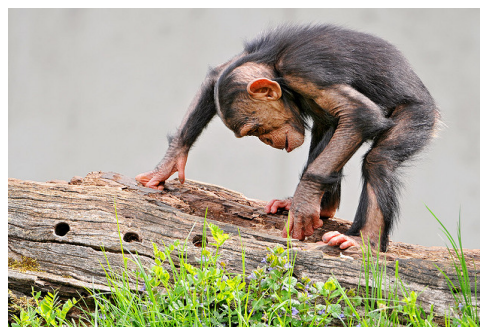
30.  $\log_{\frac{2}{5}} \frac{8}{125} = 3$

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## 2.4 Log-Arithm-etic

### *A Practice Understanding Task*

Abe and Mary are feeling good about their *log* rules and bragging about their mathematical prowess to all their friends when this exchange occurs:



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**Stephen:** I guess you think you're pretty smart because you figured out some *log* rules, but I want to know what they're good for.

**Abe:** Well, we've seen a lot of times when equivalent expressions are handy. Sometimes when you write an expression with a variable in it in a different way it means something different.

1. What are some examples from your previous experience where equivalent expressions were useful?

**Mary:** I was thinking about the *Log Logic* task where we were trying to estimate and order certain *log* values. I was wondering if we could use our log rules to take values we know and use them to find values that we don't know.

For instance: Let's say you want to calculate  $\log_2 6$ . If you know what  $\log_2 2$  and  $\log_2 3$  are then you can use the product rule and say:

$$\log_2(2 \cdot 3) = \log_2 2 + \log_2 3$$

**Stephen:** That's great. Everyone knows that  $\log_2 2 = 1$ , but what is  $\log_2 3$ ?

**Abe:** Oh, I saw this somewhere. Uh,  $\log_2 3 = 1.585$ . So Mary's idea really works. You say:

$$\log_2(2 \cdot 3) = \log_2 2 + \log_2 3$$

$$= 1 + 1.585$$

$$= 2.585$$

$$\log_2 6 = 2.585$$

2. Based on what you know about logarithms, explain why 2.585 is a reasonable value for  $\log_2 6$ ?

**Stephen:** Oh, oh! I've got one. I can figure out  $\log_2 5$  like this:

$$\begin{aligned}\log_2(2 + 3) &= \log_2 2 + \log_2 3 \\ &= 1 + 1.585 \\ &= 2.585 \\ \log_2 5 &= 2.585\end{aligned}$$

3. Can Stephen and Mary both be correct? Explain who's right, who's wrong (if anyone) and why.

Now you can try applying the *log* rules yourself. Use the values that are given and the ones that you know by definition, like  $\log_2 2 = 1$ , to figure out each of the following values. Explain what you did in the space below each question.

$$\log_2 3 = 1.585 \qquad \log_2 5 = 2.322 \qquad \log_2 7 = 2.807$$

The three rules, written for any base  $b > 1$  are:

**Log of a Product Rule:**  $\log_b(xy) = \log_b x + \log_b y$

**Log of a Quotient Rule:**  $\log_b\left(\frac{x}{y}\right) = \log_b x - \log_b y$

**Log of a Power Rule:**  $\log_b(x^k) = k \log_b x$

4.  $\log_2 9 =$  \_\_\_\_\_

5.  $\log_2 10 =$  \_\_\_\_\_

6.  $\log_2 12 =$  \_\_\_\_\_

7.  $\log_2 \left( \frac{7}{3} \right) =$  \_\_\_\_\_

8.  $\log_2 \left( \frac{30}{7} \right) =$  \_\_\_\_\_

9.  $\log_2 486 =$  \_\_\_\_\_

10. Given the work you have just done, what other values would you need to figure out the value of the base 2 log for any number?

Sometimes thinking about equivalent expressions with logarithms can get tricky. Consider each of the following expressions and decide if they are always true for the numbers in the domain of the logarithmic function, sometimes true, or never true. Explain your answers. If you answer “sometimes true”, describe the conditions that must be in place to make the statement true.

11.  $\log_4 5 - \log_4 x = \log_4 \left(\frac{5}{x}\right)$  \_\_\_\_\_

12.  $\log 3 - \log x - \log x = \log \left(\frac{3}{x^2}\right)$  \_\_\_\_\_

13.  $\log x - \log 5 = \frac{\log x}{\log 5}$  \_\_\_\_\_

14.  $5 \log x = \log x^5$  \_\_\_\_\_

15.  $2 \log x + \log 5 = \log(x^2 + 5)$  \_\_\_\_\_

16.  $\frac{1}{2} \log x = \log \sqrt{x}$  \_\_\_\_\_

17.  $\log(x - 5) = \frac{\log x}{\log 5}$  \_\_\_\_\_

## 2.4 Log-Arithm-etic – Teacher Notes

### *A Practice Understanding Task*

#### **Purpose:**

The purpose of this task is to extend student understanding of log properties and using the properties to write equivalent expressions. In the beginning of the task, students are given values of a few log expressions and asked to use log properties and known values of log expressions to find unknown values. This is an opportunity to see how the known log values can be used and to practice using logarithms and substitution. In the second part of the task, students are asked to determine if the given equations are always true (in the domain of the expression), sometimes true, or never true. This gives students an opportunity to work through some common misconceptions about log properties and to write equivalent expressions using logs.

#### **Core Standards Focus:**

**F.IF.8.** Write a function defined by an expression in different but equivalent forms to reveal and explain different properties of the function.

**F.LE.4.** For exponential models, express as a logarithm the solution to  $ab^{ct} = d$  where  $a$ ,  $c$ , and  $d$  are numbers and the base  $b$  is 2, 10, or  $e$ ; evaluate the logarithm using technology.

**Note to F.LE.4:** Consider extending this unit to include the relationship between properties of logarithms and properties of exponents, such as the connection between the properties of exponents and the basic logarithm property that  $\log(xy) = \log x + \log y$ .

#### **Standards for Mathematical Practice:**

**SMP 2 – Reason abstractly and quantitatively**

**SMP 6 – Attend to precision**

#### **The Teaching Cycle**

**Launch (Whole Class):** Launch the task by reading through the scenario and asking students to work problem #1. Follow it by a discussion of their answers, pointing out that equivalent forms often have different meanings in a story context and they can be helpful in solving equations and graphing. Follow this short discussion by having students work problems #2 and #3, then

discussing them as a class. The purpose of question #2 is to demonstrate how to use the log rules to find values, and emphasize how they can use the definition of a logarithm to determine if the value they find is reasonable. After discussing these two problems, students should be ready to use the properties to find values of logs. Have students work questions #4-10 before coming back for a discussion.

**Explore (Small Group):** As students are working, they may need support in finding combinations of factors to use so they can apply the log properties. You may want to remind them of using factor trees or a similar strategy for breaking down a number into its factors. Watch for two students that use a different combination of factors to find the value they are looking for. As you are monitoring student work, be sure they are using good notation to communicate how they are finding the values.

**Discuss (Whole Class):** Discuss a few of the problems, selecting those that caused controversy among students as they worked. For each problem, be sure to demonstrate the way to use notation and the log properties to find the values. An example might be:

$$\begin{aligned}\log_2\left(\frac{30}{7}\right) &= \log_2 30 - \log_2 7 \\ &= \log_2(5 \cdot 2 \cdot 3) - \log_2 7 \\ &= \log_2 5 + \log_2 2 + \log_2 3 - \log_2 7 \\ &= 2.322 + 1 + 1.585 - 2.807 \\ &= 2.1\end{aligned}$$

After finding each value, discuss whether or not the answer is reasonable. After a few of these problems, turn students' attention to the remainder of the task.

**Explore (Small Group):** Support students as they work in making sense of the statements and verifying them. The statements are designed to bring out misconceptions, so discussion among students should be encouraged. There are several possible strategies for verifying these equations, including using the log properties to manipulate one side of the equation to match the other or trying to put in numbers to the statement. Look for both types of strategies so that the numerical approach can provide evidence, but the algebraic approach can prove (or disprove) the statement.

**Discuss (Whole Group):** Again, select problems for discussion that have generated controversy or exposed misconceptions. It will often be useful to test the statement with numbers, although that may be difficult for students in some cases. Encourage students to cite the log property they are using as they manipulate the statements to show equivalence. For statements that are never true, ask students how they might correct the statement to make it true.

- |                 |                 |                 |
|-----------------|-----------------|-----------------|
| 11. Always true | 12. Always true | 13. Never true  |
| 14. Always true | 15. Never true  | 16. Always true |
| 17. Never true  |                 |                 |

**Aligned Ready, Set, Go: *Logarithmic Functions 2.4***

READY, SET, GO!

Name

Period

Date

**READY**

Topic: Solving simple exponential and logarithmic equations

You have solved exponential equations before based on the idea that  $a^x = a^y$ , **if and only if**  $x = y$ .

You can use the same logic on logarithmic equations.  **$\log_a x = \log_a y$ , if and only if  $x = y$**

**Rewrite each equation to set up a one-to-one correspondence between all of the parts. Then solve for  $x$ .**

| Example: <i>Original equation</i> | <i>Rewritten equation:</i> | <i>Solution:</i> |
|-----------------------------------|----------------------------|------------------|
| a.) $3^x = 81$                    | $3^x = 3^4$                | $x = 4$          |
| b.) $\log_2 x - \log_2 5 = 0$     | $\log_2 x = \log_2 5$      | $x = 5$          |

1.  $3^{x+4} = 243$

2.  $\left(\frac{1}{2}\right)^x = 8$

3.  $\left(\frac{3}{4}\right)^x = \frac{27}{64}$

4.  $\log_2 x - \log_2 13 = 0$

5.  $\log_2(2x - 4) - \log_2 8 = 0$

6.  $\log_2(x + 2) - \log_2 9x = 0$

7.  $\frac{\log 2x}{\log 14} = 1$

8.  $\frac{\log(5x-1)}{\log 29} = 1$

9.  $\frac{\log 5^{(x-2)}}{\log 625} = 1$

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**SET**

Topic: Rewriting logs in terms of known logs

**Use properties of logarithms to rewrite the indicated logarithms in terms of the given values, then find the indicated logarithm with using a calculator.**

Do not use a calculator to evaluate the logarithms.

**Given:**  $\log 16 \approx 1.2$   
 $\log 5 \approx 0.7$   
 $\log 8 \approx 0.9$

10. Find  $\log \frac{5}{8}$

11. Find  $\log 25$

12. Find  $\log \frac{1}{2}$

13. Find  $\log 80$

14. Find  $\log \frac{1}{64}$

**Given**  $\log_3 2 \approx 0.6$   
 $\log_3 5 \approx 1.5$

15. Find  $\log_3 16$

16. Find  $\log_3 108$

17. Find  $\log_3 \frac{3}{50}$

18. Find  $\log_3 \frac{8}{15}$

19. Find  $\log_3 486$

20. Find  $\log_3 18$

21. Find  $\log_3 120$

22. Find  $\log_3 \frac{32}{45}$

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GO

Topic: Using the definition of logarithm to solve for x

**Use your calculator and the definition of  $\log x$  (recall the base is 10) to find the value of x.**  
(Round your answers to 4 decimals.)

23.  $\log x = -3$

24.  $\log x = 1$

25.  $\log x = 0$

26.  $\log x = \frac{1}{2}$

27.  $\log x = 1.75$

28.  $\log x = -2.2$

29.  $\log x = 3.67$

30.  $\log x = \frac{3}{4}$

31.  $\log x = 6$

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## 2.5 Powerful Tens

### A Practice Understanding Task

### Table Puzzles

1. Find the missing values of  $x$  in the tables:

a.

| $x$ | $y = 10^x$      |
|-----|-----------------|
| -2  | $\frac{1}{100}$ |
| 1   | 10              |
|     | 50              |
|     | 100             |
| 3   | 1000            |



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b.

| $x$ | $y = 3(10^x)$ |
|-----|---------------|
|     | 0.3           |
| 0   | 3             |
|     | 94.87         |
| 2   | 300           |
|     | 1503.56       |

c. What equations could be written, in terms of  $x$  only, for each of the rows that are missing the  $x$  in the two tables above?

d.

| $x$  | $y = \log x$ |
|------|--------------|
| 0.01 | -2           |
|      | -1           |
| 10   | 1            |
|      | 1.6          |
| 100  | 2            |

e.

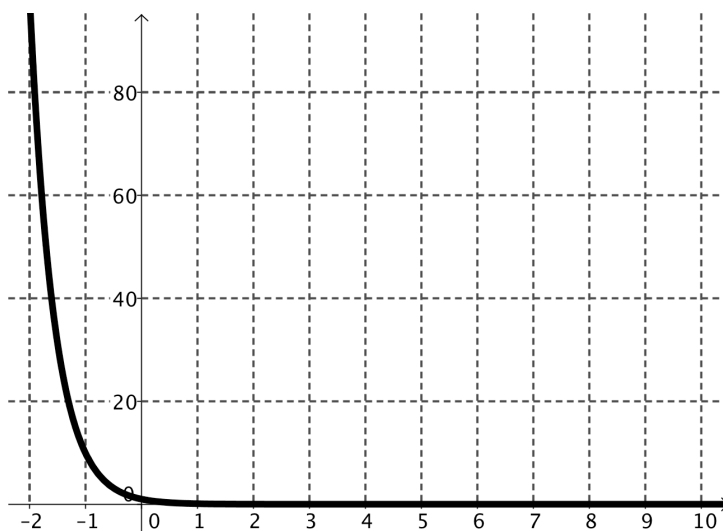
| $x$  | $y = \log(x + 3)$ |
|------|-------------------|
|      | -2                |
| -2.9 | -1                |
|      | 0.3               |
| 7    | 1                 |
|      | 3                 |

- f. What equations could be written, in terms of  $x$  only, for each of the rows that are missing the  $x$  in the two tables above?
2. What strategy did you use to find the solutions to equations generated by the tables that contained exponential functions?
3. What strategy did you use to find the solutions to equations generated by the tables that contained logarithmic functions?

## Graph Puzzles

4. The graph of  $y = 10^{-x}$  is given below. Use the graph to solve the equations for  $x$  and label the solutions.

- a.  $40 = 10^{-x}$   
 $x = \underline{\hspace{2cm}}$   
Label the solution with an A on the graph.
- b.  $10^{-x} = 10$   
 $x = \underline{\hspace{2cm}}$   
Label the solution with a B on the graph.
- c.  $10^{-x} = 0.1$   
 $x = \underline{\hspace{2cm}}$   
Label the solution with a C on the graph.



5. The graph of  $y = -2 + \log x$  is given below. Use the graph to solve the equations for  $x$  and label the solutions.

a.  $-2 + \log x = -2$

$x = \underline{\hspace{2cm}}$

Label the solution with an A on the graph.

b.  $-2 + \log x = 0$

$x = \underline{\hspace{2cm}}$

Label the solution with a B on the graph.

c.  $-4 = -2 + \log x$

$x = \underline{\hspace{2cm}}$

Label the solution with a C on the graph.

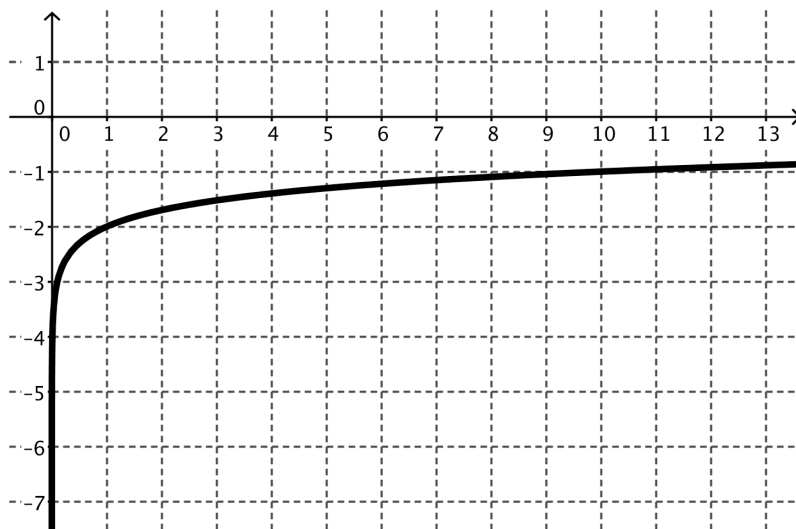
d.  $-1.3 = -2 + \log x$

$x = \underline{\hspace{2cm}}$

Label the solution with a D on the graph.

e.  $1 = -2 + \log x$

$x = \underline{\hspace{2cm}}$



6. Are the solutions that you found in #5 exact or approximate? Why?

## Equation Puzzles

Solve each equation for  $x$

7.  $10^x = 10,000$

8.  $125 = 10^x$

9.  $10^{x+2} = 347$

10.  $5(10^{x+2}) = 0.25$

11.  $10^{-x-1} = \frac{1}{36}$

12.  $-(10^{x+2}) = 16$

## 2.5 Powerful Tens – Teacher Notes

### *A Practice Understanding Task*

**Note:** Calculators or other technology with base 10 logarithmic and exponential functions are required for this task.

#### **Purpose:**

The purpose of this task is to develop student ideas about solving exponential equations that require the use of logarithms and solving logarithmic equations. The task begins with students finding unknown values in tables and writing corresponding equations. In the second part of the task, students use graphs to find equation solutions. Finally, students build on their thinking with tables and graphs to solve equations algebraically. All of the logarithmic and exponential equations are in base 10 so that students can use technology to find values.

#### **Core Standards Focus:**

**F.LE.4.** For exponential models, express as a logarithm the solution to  $ab^{ct} = d$  where  $a$ ,  $c$ , and  $d$  are numbers and the base  $b$  is 2, 10, or  $e$ ; evaluate the logarithm using technology.

#### **Standards for Mathematical Practice:**

**SMP 5 – Use appropriate tools strategically**

**SMP 6 – Attend to precision**

#### **The Teaching Cycle:**

##### **Launch (Whole Class):**

Remind students that they are very familiar with constructing tables for various functions. In previous tasks, they have selected values for  $x$  and calculated the value of  $y$  based upon an equation or other representation. They have also constructed graphs based upon having an equation or a set of  $x$  and  $y$  values. In this task they will be using tables and graphs to work in reverse, finding the  $x$  value for a given  $y$ .

### Explore (Small Group):

Monitor students as they work and listen to their strategies for finding the missing values of  $x$ . As they are working on the table puzzles, encourage them to consider writing equations as a way to track their strategies. In the graph puzzles, they will find they can only get approximate answers on a few equations. Encourage them to use the graph to estimate a value and to interpret the solution in the equation. The purpose of the tables and graphs is to help students draw upon their thinking from previous tasks to solve the equations. Remind students to connect the ideas as they work on the equation puzzles.

### Discuss (Whole Class):

Start the discussion with a student that has written and solved an equation for the third row in table b. The equation written should be:

$$94.87 = 3(10^x)$$

Ask the student to describe how they wrote the equation and then their strategy for solving it. Be sure to have students describe their thinking about how to unwind the function as the steps are tracked on the equation. Ask the class where this point would be on a graph of the function. Ask students what the graph of the function would look like, and they should be able to describe a base 10 exponential function with a vertical stretch of 3.

Move the discussion to table “e”, focusing on the last row of the table. Again, have students write the equation:

$$3 = \log(x + 3)$$

Ask the presenting student to describe his/her thinking about how to find the value of  $x$  in the table and, once again, track the steps algebraically. There are a couple of likely mistakes made by students who have tried to solve this equation algebraically. If they arise during your observation of students, discuss them here. Again, connect the solution to the graph of the function. Students should be noticing that since logs and exponentials are inverse functions, exponential equations can be solved with logs and log equations are solved with exponentials.

Move the discussion to the graph of  $y = 10^{-x}$ . Ask students to describe how they used the graph to find the solution to “a”. Ask students how they could check the solution in the equation. Does the

solution they found with the graph make sense? How would they solve this equation without a graph? Track the steps algebraically, showing something like the following:

$$40 = 10^{-x}$$

$$\log 40 = \log(10^{-x})$$

$$1.602 = -x$$

(Make sure students can explain this step, both using the calculator and simplifying the right side of the equation. It would be useful if students noticed they could use the log properties to rewrite the right side of the equation as  $-x(\log 10)$  in addition to using the definition of the logarithm.)

$$x = -1.602$$

Finally, ask students to show solutions to as many of the equation puzzles that time will allow. In every case, be sure students can describe how they use logs to undo the exponential and that their notation matches their thinking.

**Aligned Ready, Set, Go: *Logarithmic Functions 2.5***

READY, SET, GO!

Name \_\_\_\_\_

Period \_\_\_\_\_

Date \_\_\_\_\_

**READY**

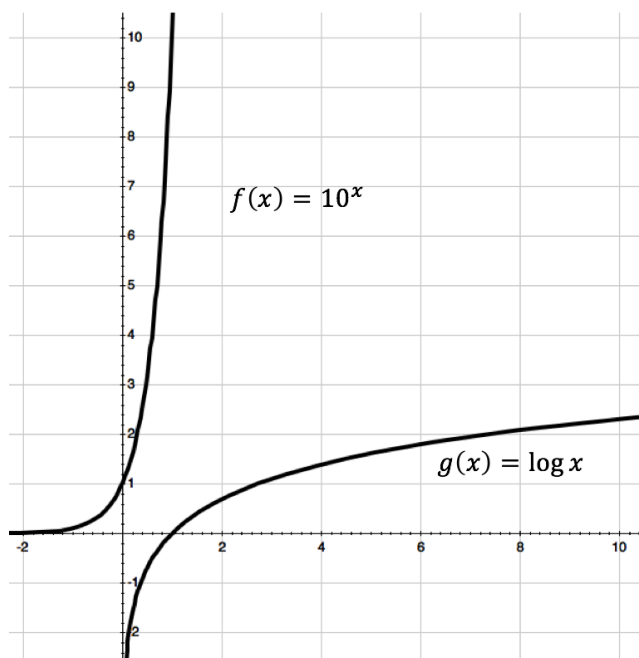
Topic: Comparing the graphs of the exponential and logarithmic functions

The graphs of  $f(x) = 10^x$  and  $g(x) = \log x$  are shown in the same coordinate plane.

Make a list of the characteristics of each function.

1.  $f(x) = 10^x$

2.  $g(x) = \log x$



Each question below refers to the graphs of the functions  $f(x) = 10^x$  and  $g(x) = \log x$ . State whether they are true or false. If they are false, correct the statement so that it is true.

- \_\_\_\_\_ 3. Every graph of the form  $g(x) = \log x$  will contain the point  $(1, 0)$ .
- \_\_\_\_\_ 4. Both graphs have vertical asymptotes.
- \_\_\_\_\_ 5. The graphs of  $f(x)$  and  $g(x)$  have the same rate of change.
- \_\_\_\_\_ 6. The functions are inverses of each other.
- \_\_\_\_\_ 7. The range of  $f(x)$  is the domain of  $g(x)$ .
- \_\_\_\_\_ 8. The graph of  $g(x)$  will never reach 3.

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### SET

Topic: Solving logarithmic equations (*base 10*) by taking the log of each side

**Evaluate the following logarithms**

- |                  |                     |                         |                  |
|------------------|---------------------|-------------------------|------------------|
| 9. $\log 10$     | 10. $\log 10^{-7}$  | 11. $\log 10^{75}$      | 12. $\log 10^x$  |
| 13. $\log_3 3^5$ | 14. $\log_8 8^{-3}$ | 15. $\log_{11} 11^{37}$ | 16. $\log_m m^n$ |

You can use this property of logarithms to help you solve logarithmic equations.

*\*Note: This property only works when the base of the logarithm matches the base of the exponent.*

**Solve the equations by inserting  $\log$  on both sides of the equation.** (You will need a calculator.)

- |                    |                    |                     |                      |
|--------------------|--------------------|---------------------|----------------------|
| 17. $10^n = 4.305$ | 18. $10^n = 0.316$ | 19. $10^n = 14,521$ | 20. $10^n = 483.059$ |
|--------------------|--------------------|---------------------|----------------------|

### GO

Topic: Solving equations involving compound interest

**Formula for compound interest:** If  $P$  dollars is deposited in an account paying an annual rate of interest  $r$  compounded (paid)  $n$  times per year, the account will contain  $A = P \left(1 + \frac{r}{n}\right)^{nt}$  dollars after  $t$  years.

21. How much money will there be in an account at the end of 10 years if \$3000 is deposited at 6% annual interest compounded as follows: (Assume no withdrawals are made.)
- a.) annually
  - b.) semiannually
  - c.) quarterly
  - d.) daily (Use  $n = 365$ .)
22. Find the amount of money in an account after 12 years if \$5,000 is deposited at 7.5% annual interest compounded as follows: (Assume no withdrawals are made.)
- a.) annually
  - b.) semiannually
  - c.) quarterly
  - d.) daily (Use  $n = 365$ .)

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## 2.6H Compounding the Problem

### *A Develop Understanding Task*



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Part I: As an enterprising young mathematician, you know that your superior knowledge of mathematics will help you make better decisions about all kinds of things in your life. One important area is money \$\$\$ . So, you've been contemplating the world and wondering how you could maximize the money that you make in your savings account.

You're young and you haven't saved much money yet. As a matter of fact, you only have \$100, but you really want to make the best of it. You like the idea of compound interest, meaning that the bank pays you interest on all the money in your savings account, including whatever interest that they had previously paid you. This sounds like a very good deal. You even remember that the formula for compound interest is exponential. Let's see, it is:

$$A = P\left(1 + \frac{r}{n}\right)^{nt}$$

Where  $A$  = the amount of money in the account at any time  $t$

$P$  = the principal, or the original amount invested in the account

$r$  = the annual interest rate

$n$  = the number of compounding periods each year

$t$  = the number of years

1. If your saving account pays a generous 5% per year and is compounded only once each year, how much money would be in the account at the end of one year?
2. How much money would be in the account at the end of 20 years?

It seems like the more compounding periods in the year, the more money that you should make. The question is, does it make a big difference?

3. Compare the amount of money that you would have after 20 years if it is compounded twice each year (semi-annually), 4 times per year (quarterly), 12 times per year (monthly), 365 times per year (daily), and then hourly. Find a way to organize, display and explain your results to your class.

It turns out that the value you found in your compounding problem is 100 times a very famous irrational number, named  $e$ . Because  $e$  is irrational it is a non-terminal, non-repeating decimal number, like  $\pi$ . The first few digits of  $e$  are 2.7182818284590452353602874713527. Like  $\pi$ ,  $e$  is a number that occurs in the mathematics of many real world situations, including exponential growth. One of the formulas using  $e$  results from the thinking that you just did about compound interest. It can be shown that the amount of money  $A$  in a savings account where money is compounded continuously is given by:

$$A = Pe^{rt}$$

$P$  = the principal, or the original amount invested in the account

$r$  = the annual interest rate

$t$  = the number of years

4. It is fairly typical for savings accounts to be compounded monthly. Compare the amount of money in two savings accounts after 10 years with the same initial investment of \$500 and interest rate of 3% in each account if the first account is compounded monthly and the second account is compounded continuously.

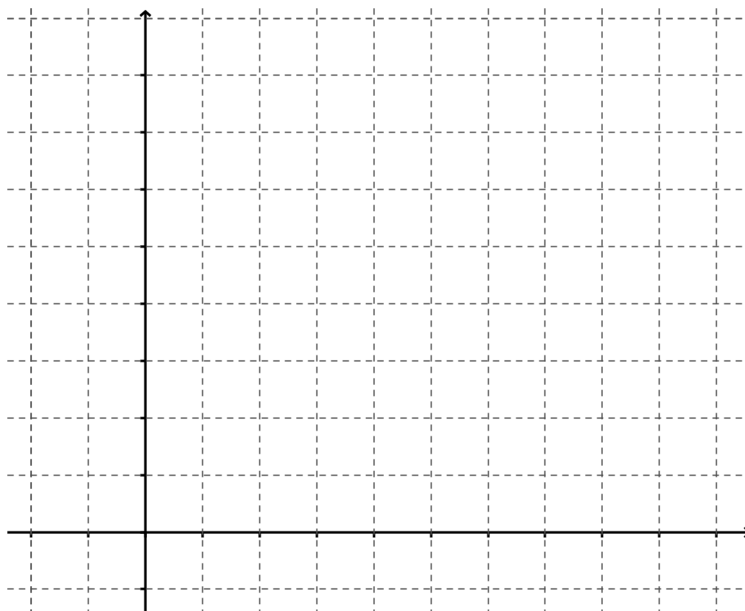
5. Use technology to compare the graphs of the two accounts. What conclusions would you draw about the effect of changing the number of compounding periods on a savings account?

## Part II

Since  $e$  is widely used to model exponential growth and decay in many contexts, let's get a little more familiar with the base  $e$  exponential function:

$$f(x) = e^x$$

1. Make a prediction about the graph of  $f(x)$ . Explain what knowledge you used to make your prediction.
2. Create a table and a graph and describe the mathematical features of  $f(x)$ .



## 2.6H Compounding the Problem – Teacher Notes

### *A Develop Understanding Task*

#### **Purpose:**

The purpose of this task is to introduce students to the irrational number,  $e$ . The task begins with students considering how much is earned in a savings account earning compound interest as the number of compounding periods increases. The amount in the account begins to converge to the value of  $100e$ . The formula for continuous compound interest is given and compared to monthly compounding.

#### **Core Standards Focus:**

**F.IF.7.a** Graph exponential and logarithmic functions, showing intercepts and end behavior, and trigonometric functions, showing period, midline, and amplitude.

#### **Standards for Mathematical Practice:**

**SMP 2 – Reason abstractly and quantitatively**

**SMP 5 – Use appropriate tools strategically**

#### **Vocabulary: Continuous compound interest**

#### **The Teaching Cycle:**

##### **Launch (Whole Class):**

Begin the task by reminding students about the idea of compound interest (which was explored in Secondary Mathematics I) and the variables in the formula. Model using the formula with problem 1 and then ask students to do problem 2 on their own. Check that students got the correct answer of \$265.33 by using the formula and calculator (or other technology) correctly. Then ask students to work on problem 3.

##### **Explore (Small Group):**

As students are working encourage them to organize their results so they can be displayed. Look for students that have created tables and graphs. Students may have some trouble with getting a good window for a graph or confusion about why it looks as it does. This can be part of the class discussion. Since the idea of #3 is to show that as the number of compounding periods increases

the value of the expression,  $(1 + \frac{r}{n})^{nt}$  converges to  $e$ , ask students to keep several decimal places in their answers.

**Discuss (Whole Class) :**

Part I: Begin the discussion with a table such as the one below:

| Number of Compounding Periods Per Year | \$ Amount in the Account |
|--|--------------------------|
| 1                                      | 265.329770514442         |
| 2                                      | 268.506383838999         |
| 4                                      | 270.148494075333         |
| 12                                     | 271.264028548199         |
| 365                                    | 271.809566814756         |
| 8760                                   | 271.827407082761         |
| 525600                                 | 271.828169861769         |

Ask students what they notice about the table. What would the graph of this function look like based on the table? If students have tried to create a graph, ask them to show their work. If not, ask students how they could see the function graphically using the formula. Use technology to graph:

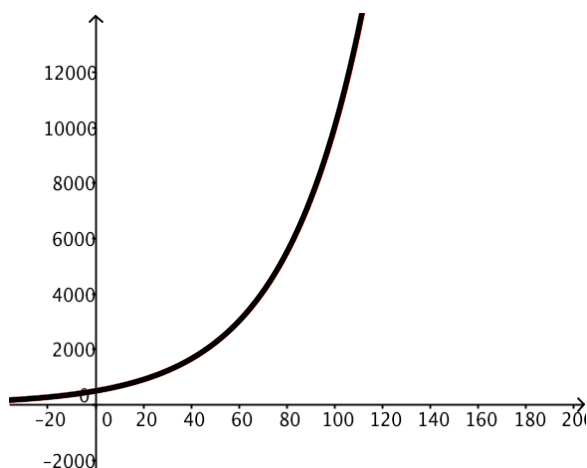
$y = 100(1 + \frac{.05}{x})^{20x}$ . You will probably need to discuss how to use the work done to create the table leads to this function. Depending on the window selected this graph may look like a horizontal line. Ask students to interpret what they see. Is the graph actually horizontal? Trace along the graph to show that is actually increasing very slightly as it converges toward  $100e$ .

At this point, explain about the irrational number,  $e$ . Ask students to complete the task.

Part II:

Discuss questions 4 and 5. The graph for #5 is below. Notice that the two graphs are nearly the same. Trace along the curves so students can see the difference between the two functions. Ask students what conclusions they would draw about the importance of the number of compounding periods in the investment? If the number of compounding periods doesn't make much difference, what variables do make a big difference for a given amount of principal? This is a good time for students to notice that since the amount in the account is growing exponentially, the amount

doesn't grow much in the first few years, but takes off after a number of years. This is why financial advisors always suggest that we begin investing early.



Wrap up the lesson with a discussion of the features of  $f(x) = e^x$ . Be sure to highlight how this function is like other exponential functions that students have learned about. Stress the importance of  $e$ . Just as  $\pi$  occurs naturally as the ratio of the diameter to the circumference in a circle,  $e$  occurs naturally in continuous growth situations. The next task will give more opportunities to work with continuous growth.

**Aligned Ready, Set, Go: *Logarithmic Functions 2.6H***

READY, SET, GO!

Name

Period

Date

## READY

Topic: Identifying linear and quadratic patterns by examining the rates of growth

**Use first and second differences to identify the pattern in the tables as *linear*, *quadratic*, or *neither*. Write the recursive equation for the patterns that are *linear* or *quadratic*.**

1.

| $x$ | $y$ |
|-----|-----|
| -3  | -23 |
| -2  | -17 |
| -1  | -11 |
| 0   | -5  |
| 1   | 1   |
| 2   | 7   |
| 3   | 13  |

a. pattern

b. recursive equation

2.

| $x$ | $y$ |
|-----|-----|
| -3  | 4   |
| -2  | 0   |
| -1  | -2  |
| 0   | -2  |
| 1   | 0   |
| 2   | 4   |
| 3   | 10  |

a. pattern

b. recursive equation

3.

| $x$ | $y$ |
|-----|-----|
| -3  | -15 |
| -2  | -10 |
| -1  | -5  |
| 0   | 0   |
| 1   | 5   |
| 2   | 10  |
| 3   | 15  |

a. pattern

b. recursive equation

4.

| $x$ | $y$ |
|-----|-----|
| -3  | 24  |
| -2  | 22  |
| -1  | 20  |
| 0   | 18  |
| 1   | 16  |
| 2   | 14  |
| 3   | 12  |

a. pattern

b. recursive equation

5.

| $x$ | $y$ |
|-----|-----|
| -3  | 48  |
| -2  | 22  |
| -1  | 6   |
| 0   | 0   |
| 1   | 4   |
| 2   | 18  |
| 3   | 42  |

a. pattern

b. recursive equation

6.

| $x$ | $y$ |
|-----|-----|
| -3  | 4   |
| -2  | 1   |
| -1  | 0   |
| 0   | 1   |
| 1   | 4   |
| 2   | 9   |
| 3   | 16  |

a. pattern

b. recursive equation

## SET

Topic: Compounding interest continuously and base  $e$

**Recall the equations for compound interest you used in class today:**

$$A = P \left( 1 + \frac{r}{n} \right)^{nt} \text{ and } A = Pe^{rt}$$

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7. Calculate the amount of money in an account after 15 years if \$7000 is deposited at 6% annual interest compounded as follows:
- a) annually    b) semiannually    c) quarterly    d) daily (Use  $n = 365$ .)    e) continuously
8. How much money will be in an account at the end of 34 years if \$17,000 is deposited at 12% annual interest compounded as follows?
- a) annually    b) semiannually    c) quarterly    d) daily (Use  $n = 365$ .)    e) continuously
9. Fill in the table for each of the given functions. Then graph each function on the same axes.

a)

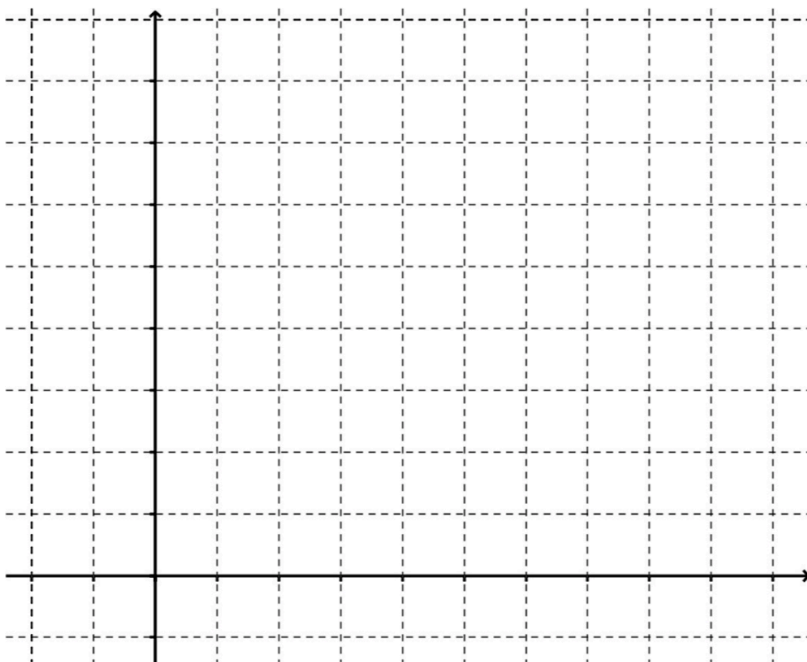
| x  | $f(x) = 2^x$ |
|----|--------------|
| -2 |              |
| -1 |              |
| 0  |              |
| 1  |              |
| 2  |              |

b)

| x  | $g(x) = 4^x$ |
|----|--------------|
| -2 |              |
| -1 |              |
| 0  |              |
| 1  |              |
| 2  |              |

c)

| x  | $h(x) = e^x$ |
|----|--------------|
| -2 |              |
| -1 |              |
| 0  |              |
| 1  |              |
| 2  |              |



10. What point do all 3 functions share?

Why?

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11. Given that  $f(x) = 2^x$ ,  $g(x) = 4^x$ , and  $h(x) = e^x$ , create a true inequality by filling in the spaces on the inequalities with  $f(x)$ ,  $g(x)$ , or  $h(x)$ .

a) \_\_\_\_\_ < \_\_\_\_\_ < \_\_\_\_\_ when  $x > 0$

b) \_\_\_\_\_ < \_\_\_\_\_ < \_\_\_\_\_ when  $x < 0$

c) Write an expression that describes the relationship between  $f(x)$ ,  $g(x)$ , and  $h(x)$  when  $x = 0$ .

### GO

Topic: Exploring properties of logarithms

**Replace the question marks and fill in the blanks using your own logic and what you know about logarithms.**

12.  $\log_7 1 = 0$  because  $7^? =$  \_\_\_\_\_. 13.  $\log_a 1 = 0$  because  $a^? =$  \_\_\_\_\_.

14.  $\log_6 6 = 1$  because  $6^? =$  \_\_\_\_\_. 15.  $\log_a a = 1$  because  $a^? =$  \_\_\_\_\_.

16.  $\log_4 4^x =$  \_\_\_\_\_ because  $4^? = 4^x$ . 17.  $\log_a a^x =$  \_\_\_\_\_ because  $a^? = a^x$ .

18. If  $\log_a x = \log_a y$ , then \_\_\_\_\_ = \_\_\_\_\_.

19.  $3^{\log_3 81} =$  \_\_\_\_\_ because  $\log_3 81$  equals the exponent \_\_\_\_\_ that makes  $3^? = 81$ .

20.  $a^{\log_a x} =$  \_\_\_\_\_ because  $\log_a x$  equals the exponent that makes  $a^? = x$ .

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## 2.7H Logs Go Viral

### *A Solidify Understanding Task*

As we learned in *Compounding the Problem*, when money is compounded continuously, it turns out that the base of the exponential growth function is  $e \approx 2.71828$ . It is true that basically all situations that grow or decay continuously can be modeled with a base  $e$  exponential function. The basic formulas are:

Continuous growth:

$$A = Pe^{rt}$$

Continuous Decay:

$$A = Pe^{-rt}$$

In both cases,  $A$  is the amount of “stuff” at time  $t$ .  $P$  is the amount of “stuff” to start with, or at  $t = 0$ , and  $r$  is the rate of growth or decay. So, if we were talking about money,  $A$  and  $P$  are dollar amounts. If we’re talking about a population,  $A$  and  $P$  are the number of people in the population. If we’re talking about radioactive decay, then  $A$  and  $P$  are the mass of the radioactive substance. In other words,  $P$  is what we started with and  $A$  is what we have after it grows or decays.

Since there are many things that grow or decay continuously in nature,  $e$  is called the natural exponential function and the base  $e$  logarithm is called the natural logarithm. Think about the growth of bacteria. One cell splits into two cells. The two cells begin to split, and the resulting cells split and so on. In a group of many cells, there would be a cell splitting nearly continuously, thus it is a base  $e$  exponential function. So, to use this base  $e$  exponential and its inverse, imagine the following scenarios:

You are an epidemiologist, a person who studies the outbreak and spread of diseases. Part of your job is to help avoid a pandemic—a worldwide outbreak of a disease. You know some of the most difficult diseases to deal with are viruses because they don’t respond to many of the medicines we have available and because viruses are able to mutate and change quickly, making it more difficult to contain them.

You have been studying a new virus that causes people to break out in spots. Suddenly, a colleague rushes into your office to inform you there is a confirmed outbreak of the virus in Europe. The growth of the virus through a population is continuous (until it is somehow contained) at a rate around 3% per day. The current outbreak has 5 confirmed victims.



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1. Using this information, create a model of the spread of the spotted virus in this region if it is not contained. To simplify your model slightly, consider the 5 victims as the number of victims on day 0.

Table

Graph

Equation

2. Based on your model, how many people will be infected on:
  - a) Day 30?
  - b) Day 60?
3. Based upon your model, on what day will there be 50 victims? Show how you arrived at your answer.

4. Will the number of days it will take the virus to claim 100 victims be double the number of days that it took to claim 50 victims? Why or why not?

5. Calculate the number of days it will take for the virus to claim 100 victims.

6. On what day will there be 150 victims?

Now you have received a report of a mysterious illness that seems to turn the infected humans into mindless zombies has broken out in a major American city. Since the hungry zombies prey upon innocent people, the outbreak grows continuously at a rate of 12% per day. The outbreak begins with 80 people.

7. How many zombies will there be after 5 days?

8. How many days will it take for the zombie population to reach 3,700,000 (about the population of Los Angeles, CA)?

9. At what rate would the zombie population be growing if it reached 190,000 people (about the population of Salt Lake City, Utah) in 20 days?

Now we're going to get a little more far-fetched in the scenario. Let's say that zombies produce radioactive goo that decays continuously with a half-life of 3 years. (That's one more danger of having zombies around.) The half-life tells us that after 3 years, only half of the amount of goo we started with is remaining.

10. If we start with 10 pounds of zombie goo, how much will be remaining after 5 years?
11. How long will it take for the amount of zombie goo to decay to an amount less than 0.5 pounds?
12. When will there be no zombie goo left?

## 2.7H Logs Go Viral – Teacher Notes

### *A Solidify Understanding Task*

**Purpose:** The purpose of this task is for students to understand continuous growth and decay. Students are introduced to the formulas for growth and decay and asked to solve problems using the formulas. The problems will require thinking about the inverse of the base  $e$  exponential function, the natural logarithm. Students should understand the concept of a logarithm at this point, but will need to be introduced to the notation for natural log ( $\ln x$ ).

**Core Standards Focus: F.LE.4.** For exponential models, express as a logarithm the solution to:  $ab^{ct} = d$  where  $a$ ,  $c$ , and  $d$  are numbers and the base  $b$  is 2, 10, or  $e$ ; evaluate the logarithm using technology.

**Related Standards: F.LE.3**

**Standards for Mathematical Practice:**

**SMP 4 – Model with mathematics**

**SMP 6 – Attend to precision**

**Vocabulary:** natural logarithm

**The Teaching Cycle:**

**Note:** This task requires access to technology such as graphing calculators.

**Launch (Whole Class):** Begin the task by explaining about continuous growth and decay and the use of the base  $e$  exponential, as described in the first paragraphs of the task. Be sure students understand the variables in the formulas and that  $e$  is the constant they learned about in the previous task. (When students first learn about  $e$ , they often have to be reminded that it is not a variable.) Ask students to work #1a on their own. When they have had some time to use the formula to write an equation and model the spread of the virus, discuss the model with the class. Ask why the growth appears slow in the beginning and then grows more quickly, ensuring that students connect to the idea that the rate of growth depends on the number of people infected. The

rate of continuous growth generally depends on the amount of “stuff” that is growing in any context.

**Explore (Small Group):** Let students work on the task, monitoring their discussion. Listen for how students are thinking about “undoing” the base  $e$  exponential. They should be talking about a base  $e$  logarithm. When most students arrive at this point (after problem 4), you may wish to stop and discuss the proper notation for natural log with the class.

**Discuss (Whole Class):** Begin the discussion with #7. Ask a student to describe how they used the formula and demonstrate how to use technology to find a value. Then ask a student to describe the steps they used to solve the equation in #8. Record the steps as they demonstrate their work and support their use of the natural log notation. The steps may be something like this:

1. Use the formula to write the equation:  $3,700,000 = 80e^{0.12t}$
2. Divide both sides by 80.
3. Take the natural log of both sides.
4. Divide by 0.12

Ask a student to show how they solved the last problem. If time permits, use technology to model the decay of the goo, and ask students how they could verify their solution graphically. Ask if the zombie goo will ever be entirely gone, using both the graph and an equation to support their answers.

Ask how solving these equations is similar to solving the base 10 equations they solved in Powerful Tens.

**Aligned Ready, Set, Go: *Logarithmic Functions 2.7H***

READY, SET, GO!

Name \_\_\_\_\_

Period \_\_\_\_\_

Date \_\_\_\_\_

## READY

Topic: Identifying polynomial patterns by examining the rates of growth

Fill in the table for each of the given functions.

1.

| $x$ | $y = x^1$ |
|-----|-----------|
| -3  |           |
| -2  |           |
| -1  |           |
| 0   |           |
| 1   |           |
| 2   |           |
| 3   |           |
|     |           |

2.

| $x$ | $y = x^2$ |
|-----|-----------|
| -3  |           |
| -2  |           |
| -1  |           |
| 0   |           |
| 1   |           |
| 2   |           |
| 3   |           |
|     |           |

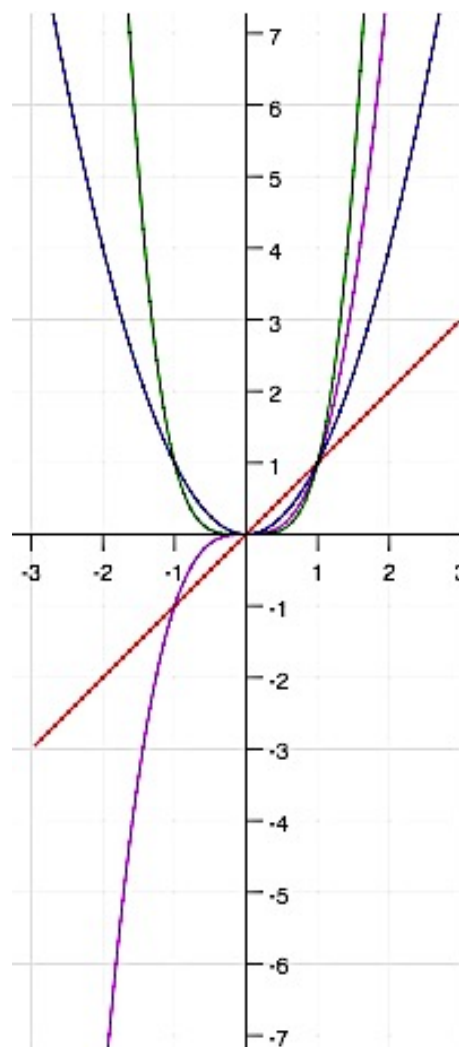
3.

| $x$ | $y = x^3$ |
|-----|-----------|
| -3  |           |
| -2  |           |
| -1  |           |
| 0   |           |
| 1   |           |
| 2   |           |
| 3   |           |

4.

| $x$ | $y = x^4$ |
|-----|-----------|
| -3  |           |
| -2  |           |
| -1  |           |
| 0   |           |
| 1   |           |
| 2   |           |
| 3   |           |

5. Label each graph with the function that describes it.



6. Identify the point(s) that all of the functions share.

Explain why this is logical.

7. Contrast the graphs of the even functions with the graphs of the odd functions.

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**SET**

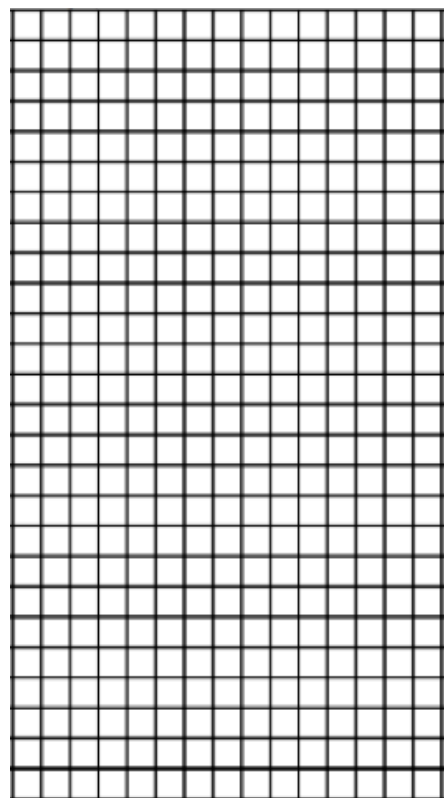
Topic: Using logarithms to solve exponential equations

8. A certain bacteria population is known to double every 15 minutes. An experiment is being conducted in a microbiology lab. Suppose there are initially 7 bacteria in a petri dish. Make a table, graph, and an equation that will predict the number of bacteria in  $t$  hours. Label the scale on both the x and y axes. The increments on the x-axis should be  $\frac{1}{4}$  of an hour or less. Make sure you can fit at least 3 points on your graph.

| Time in hours<br>(4 periods of<br>doubling per hour) | Number of<br>bacteria |
|--|-----------------------|
| 0  |                       |
| 1  |                       |
| 2  |                       |
| 3  |                       |
| 4  |                       |
| 5  |                       |

Equation:

9. Between what times to the nearest  $\frac{1}{4}$  of an hour will the number of bacteria exceed 10,000? 1,000,000?
10. Predict the number of bacteria after a 24 hour period. (Write your answer in scientific notation)
11. Write a logarithmic equation that would allow you to find the time  $t$  when there are 700 bacteria.
12. Calculate the time when there are 700 bacteria. (Round your answer to 3 decimals.)



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GO

Topic: Applying the properties of logarithms

**Use the properties of logarithms and the values below to find the value of the indicated logarithm. Do not use a calculator to evaluate the logarithms.**

13.  $\log 12 \approx 1.1$  Find  $\log \frac{2}{3}$ .

$$\log 8 \approx 0.9$$

$$\log 7 \approx 0.8$$

14.  $\log 12 \approx 1.1$  Find  $\log \frac{1}{7}$ .

$$\log 8 \approx 0.9$$

$$\log 7 \approx 0.8$$

15.  $\log 12 \approx 1.1$  Find  $\log \frac{7}{8}$ .

$$\log 8 \approx 0.9$$

$$\log 7 \approx 0.8$$

16.  $\log 12 \approx 1.1$  Find  $\log \frac{3}{14}$ .

$$\log 8 \approx 0.9$$

$$\log 7 \approx 0.8$$

17.  $\log_8 6 \approx 0.86$  Find  $\log_8 729$ .

$$\log_8 9 \approx 1.06$$

$$\log_8 7 \approx 0.94$$

18.  $\log_8 6 = A$  Find  $\log_8 729$ .

$$\log_8 9 = B$$

$$\log_8 7 = C$$

19.  $\log_8 6 \approx 0.86$  Find  $\log_8 \frac{2}{3}$ .

$$\log_8 9 \approx 1.06$$

$$\log_8 7 \approx 0.94$$

20.  $\log_8 6 = A$  Find  $\log_8 \frac{14}{3}$ .

$$\log_8 9 = B$$

$$\log_8 7 = C$$

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## 2.8H Choose This, Not That

### *A Solidify Understanding Task*

In each of the following equations you are given two options for the next step. Your job is to pick the most productive of the two options, solve the equation and check your solution to be sure you made the right choice. When you are finished, go back and explain why the option you did not choose was either wrong or unproductive.

1.  $\log 2x = 3$

Option 1:  $2x = \log 3$

Option 2:  $10^3 = 2x$

Solution:

Check:

Why I didn't select Option \_\_\_\_:

2.  $\ln(x + 3) = 2$

Option 1:  $\ln x + \ln 3 = 2$

Option 2:  $e^2 = x + 3$

Solution:

Check:

Why I didn't select Option \_\_\_\_:



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3.  $\log_3(2x + 1) = 2$

Option 1:  $3^{2x+1} = 3^2$

Option 2:  $2x + 1 = 3^2$

Solution:

Check:

Why I didn't select Option \_\_\_\_:

4.  $\log_5(2x - 7) = \log_5 3$

Option 1:  $2x - 7 = 3$

Option 2:  $5^3 = 2x - 7$

Solution:

Check:

Why I didn't select Option \_\_\_\_:

5.  $2 \log_3 x = \log_3 4$

Option 1:  $2x = 4$

Option 2:  $\log_3 x^2 = \log_3 4$

Solution:

Check:

Why I didn't select Option \_\_\_\_:

6.  $3 \ln x = \ln 16 + \ln 4$

Option 1:  $\ln x^3 = \ln(16 \cdot 4)$

Option 2:  $3x = 16 + 4$

Solution:

Check:

Why I didn't select Option \_\_\_\_:

7.  $\log_2 2x - \log_2(x - 2) = \log_2 3$

Option 1:  $\log_2 \left( \frac{2x}{x-2} \right) = \log_2 3$

Option 2:  $\frac{\log_2 2x}{\log_2(x-2)} = \log_2 3$

Solution:

Check:

Why I didn't select Option \_\_\_\_:

8.  $-2 = \log_x \frac{1}{9}$

Option 1:  $x^{-2} = \frac{1}{9}$

Option 2:  $-2 = \log_x 1 - \log_x 9$

Solution:

Check:

Why I didn't select Option \_\_\_\_:

9.  $x = \log_3 10$

Option 1:  $x^3 = 10$

Option 2:  $3^x = 10$

Solution:

Check:

Why I didn't select Option \_\_\_\_:

10.  $\log_a(x^2 + 1) + 2\log_a 4 = \log_a 40x$

Option 1:  $\log_a 16(x^2 + 1) = \log_a 40x$

Option 2:  $\log_a 8(x^2 + 1) = \log_a 40x$

Solution:

Check:

Why I didn't select Option \_\_\_\_:

## 2.8H Choose This, Not That – Teacher Notes

### *A Solidify Understanding Task*

**Purpose:** The purpose of this task is for students to become more skilled in using the notation for logarithms and solving equations with logs. In each case, students are given the choice of two different steps to take, leading to a solution to the equation. Many of the common misconceptions in using properties of logarithms are offered as choices so students have an opportunity to better understand the properties of logarithms.

#### **Core Standards Focus:**

This task extends the standards to ensure students are prepared to use logarithm properties and solve equations involving logarithms.

#### **Related Standards: F.LE.4**

#### **Standards for Mathematical Practice:**

**SMP 1 – Make sense of problems and persevere in solving them**

**SMP 7 – Look for and make use of structure**

#### **The Teaching Cycle:**

**Launch (Whole Class):** Begin the task by making sure students understand the instructions for the task and reviewing the properties of logarithms. Tell students they will be working with equations that use logarithms in several different ways, and they should be prepared to use log properties and definitions to solve the equation. Give students enough time to try several of the choices on their own before switching to groups or pairs to discuss their choices.

**Explore (Small Group):** While students are working together, watch for disagreements about options that highlight common misconceptions or misapplications of the log rules. Also listen for students discussing how to “get rid of the logs” in an equation. Are students recognizing the use of the definition of a logarithm, the equality property of logarithms, or simply thinking that they go away? Be prepared to press this point during the discussion.

**Discuss (Whole Class):** Ask students to present their work, including their reasons for eliminating one of the options, for as many equations as possible. Emphasize the application of log

properties as they arise. After working some problems, highlight the difference between problem 1 and problem 4. Ask students what definition or property allows them to eliminate the logarithm in each problem. They should be familiar with using the definition of logarithms to evaluate logs from earlier work, but they may not recognize how they are using it in problem 1. In problem 4, students may reason that if the two log expressions are equal, then the arguments of the logs must also be equal. This allows us to “drop” the logarithms. Tell students this idea is the basis for the equality property of logarithms:

$$\text{If } \log_a x = \log_a y, \text{ then } x = y$$

This property is used in several problems, but students often want to misapply the property by just dropping the logs off of both sides of the equation without simplifying to only one log expression on each side so the property of equality applies. This point is first highlighted in #6 and is also available in several other problems. Students should recognize the need to use other log properties to combine expressions before using the equality property or the definition of a logarithm to solve the equation.

Problem 9 will be difficult for students to find a solution, although they should recognize the correct first step. During the discussion of this problem, show students that after the first step, it would be useful to take the log of both sides, but taking a base 3 log will not help to get a solution.

Demonstrate how to take either the natural log or base 10 log of each side of the equation and use the result to find a solution, both exact and approximate. This becomes an opportunity to show a similar strategy would work for any exponential in the same form and leads to the change of base formula:

$$\log_a m = \frac{\log m}{\log a} = \frac{\ln m}{\ln a}.$$

This formula works for any base and is particularly useful for finding approximate values for log expressions and graphing using technology.

In addition to the two new properties demonstrated in these problems, be sure that by the end of the discussion students have two important ideas about solving log and exponential equations:

1. It is generally helpful to arrange the equation so there is only one term on each side. Log properties are often used to combine terms so this can be achieved.

2. When there is only one term on each side, use the inverse operation. If the equation is exponential, undo it using a log. If the equation is logarithmic, undo it using an exponential. If there are logs on both sides, set the arguments equal.

Students will have more practice using these principles in the next task.

**Aligned Ready, Set, Go: *Logarithmic Functions 2.8H***

READY, SET, GO!

Name \_\_\_\_\_

Period \_\_\_\_\_

Date \_\_\_\_\_

### READY

Topic: Evaluating functions

1. Find  $h(-11)$  given that  $h(x) = 2x^2 + 9x - 43$
2. Find  $r(-1)$  given that  $r(x) = -5x^2 - 3x + 9$
3. Find  $f(4)$ , given that  $f(x) = x^2 + 11$ .
4. Find  $m(3)$  given that  $m(x) = \log_x 81$ .
5. Find  $g(-3)$  given that  $g(x) = x^2 + 2x + 4$ .
6. Find  $p(3)$  given that  $p(x) = 5^x + 2x$ .
7. Find  $q(2)$  given that  $q(x) = 7^x + 11x$ .
8. Find  $s\left(\frac{1}{2}\right)$  given that  $s(x) = 12x^2$ .

### SET

Topic: Finding solutions to logarithmic equations

**Three possible solutions are given for each equation. Determine which solution is correct.**

**Justify your answers.**

- |  |  |   |
|--|--|---|
| <p>9. <math>\log 5x = 3</math></p> <p>a. <math>x = 3 - \log 5</math></p> <p>b. <math>x = 200</math></p> <p>c. <math>x = \frac{3}{5}</math></p>                           | <p>10. <math>\log(x + 28) = 2</math></p> <p>a. <math>x = 72</math></p> <p>b. <math>x = 2 - \log 28</math></p> <p>c. <math>x = \frac{100}{28}</math></p>              | <p>11. <math>\log_3(2x + 1) = 2</math></p> <p>a. <math>x = \frac{1}{2}</math></p> <p>b. <math>x = \frac{5}{2}</math></p> <p>c. <math>x = 4</math></p> |
| <p>12. <math>\log_5(3x - 8) = \log_5 13</math></p> <p>a. <math>x = 7</math></p> <p>b. <math>x = \frac{5^{13} + 8}{3}</math></p> <p>c. <math>x = \frac{104}{3}</math></p> | <p>13. <math>3\log x = \log 16 + \log 4</math></p> <p>a. <math>x = \frac{20}{3}</math></p> <p>b. <math>x = 4</math></p> <p>c. <math>x = \sqrt[3]{10^{20}}</math></p> | <p>14. <math>\log_2 2x - \log_2(x - 2) = \log_2 3</math></p> <p>a. <math>x = 6</math></p> <p>b. <math>x = 3</math></p> <p>c. <math>x = -6</math></p>  |
| <p>15. <math>-3 = \log_x\left(\frac{1}{8}\right)</math></p> <p>a. <math>x = -2</math></p> <p>b. <math>x = 2</math></p> <p>c. <math>x = 4</math></p>                      | <p>16. <math>x = \log_3 15</math></p> <p>a. <math>x = \sqrt[3]{15}</math></p> <p>b. <math>x = 5</math></p> <p>c. <math>x \approx 2.465</math></p>                    | <p>17. <math>\log_a(x - 7) = 0</math></p> <p>a. <math>x = 7</math></p> <p>b. <math>x = 8</math></p> <p>c. no solution</p>                             |

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**Circle the expressions that are equal. Explain why they are equal.**

18.  $\log_5 \sqrt{50}$ ,  $\log_5 25$ ,  $1 + \log_5 \sqrt{2}$

19.  $\frac{\log_2 32}{\log_2 4}$ ,  $\log_2 \frac{32}{4}$ ,  $\log_2 32 - \log_2 4$

20.  $\log \sqrt{90}$ ,  $\log 3 + \frac{1}{2}$ ,  $\frac{1}{2} \log 2 + \log 45$

21.  $\log_7 \left( \frac{1}{49} \right)$ ,  $\log_7 1 - \log_7 49$ ,  $-2(\log_7 7)$

**GO**

Topic: Solving exponential equations

**Solve for x.**

22.  $4^{(2x-7)} = 64$

23.  $5^x = \frac{1}{125}$

24.  $3^{(2x+8)} = 729$

25.  $\left( \frac{1}{2} \right)^x = 128$

26.  $36^{(x+5)} = 216^{(x-3)}$

27.  $\left( \frac{2}{3} \right)^x = \frac{16}{81}$

28.  $3^{-x} = 27$

29.  $\left( \frac{3}{4} \right)^x = \frac{16}{9}$

30.  $125^{(3x-4)} = 625^{(x+1)}$

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## 2.9H Don't Forget Your Login

### *A Practice Understanding Task*

Solve each of the following equations. When you have finished, sort the equations into categories based upon the strategy you used to solve them. Name each category and then describe how to solve equations in this category.

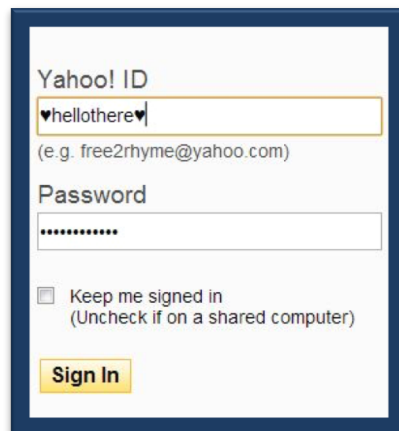
1.  $\log 3x = 2$

2.  $-3 = \log_x \frac{1}{125}$

3. The rate at which caffeine is eliminated from the bloodstream of an average adult is about 15% per hour. If the peak level of caffeine in the bloodstream is 30 milligrams, the amount of caffeine left in the bloodstream  $t$  hours after the caffeine reaches its peak level can be modeled by the function:  $C(t) = 30(0.85)^t$ . After how many hours will there be 15 mg left in the bloodstream?

4.  $x = \log_5 100$

5.  $\ln(5x - 3) + \ln 2 = \ln(24 - 2x)$



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<https://flic.kr/p/dTcZhM>

6.  $\log_5(4x - 3) = \log_5 29$
7. The Richter scale, which measures the magnitude of earthquakes is a logarithmic scale, where the magnitude of the earthquake,  $M$  depends on the energy released by the earthquake  $E$  (in Joules). In 1994, an earthquake of magnitude 6.6 on the Richter scale injured thousands of people and cost billions of dollars in damages. That earthquake could be modeled with the equation:  $6.6 = \frac{2}{3} \log \left( \frac{E}{10^{11.8}} \right)$ . Find the energy released by the earthquake.
8.  $\log_5(3x + 1) = 2$
9.  $\log_b x^3 = \log_b 27$
10. Ever wonder why suddenly your kitchen is full of fruit flies? Given good conditions, fruit fly populations can grow at the amazing rate of 28% per day. If 25 fruit flies enter your house to hang out on a piece of ripe fruit, the fly population after  $t$  days can be modeled as:  $P(t) = 25(1.28)^t$ . How long will it take for you to have 100 little fruit flies buzzing around?
11.  $\log_x 5 = \frac{1}{4}$
12.  $3^x = 5^{2.3}$

13.  $\log_2 2x - \log_2(x - 2) = \log_2 3$

14.  $\log_3 2x = \log_3(x - 1)$

15.  $\ln(x - 1) = 3$

16.  $\log(x^2 - 2) + 2 \log 6 = \log 6x$

17.  $x = \log_3 10$

18.  $2\log_a x + \log_a 2 = \log_a(5x + 3)$

19.  $3 + 7^{3x+1} = 346$

|  |  |
|--|--|
| Category Name and Description:           | Category Name and Description:           |
| How to solve equations in this category: | How to solve equations in this category: |
| Category Name and Description:           | Category Name and Description:           |
| How to solve equations in this category: | How to solve equations in this category: |
| Category Name and Description:           | Category Name and Description:           |
| How to solve equations in this category: | How to solve equations in this category: |

Note: You may not need all these categories.

## 2.9H Don't Forget Your Login – Teacher Notes

### *A Practice Understanding Task*

**Purpose:** The purpose of this task is for students to develop fluency with solving exponential and logarithmic equations. In the task students are asked to solve a number of equations (some given as formulas in a context), categorize the various types of equations, and then write a procedure for solving each type of equation.

#### **Core Standards Focus:**

This task extends the standards to ensure students are prepared to use logarithm properties and solve equations involving logarithms.

#### **Related Standards: F.LE.4**

#### **Standards for Mathematical Practice:**

**SMP 6 – Attend to precision**

**SMP 8 – Look for and express regularity in repeated reasoning**

**Vocabulary:** Extraneous solution

#### **The Teaching Cycle:**

**Launch (Whole Class):** Begin the task by asking students about some of the strategies for solving log and exponential equations that surfaced in the previous tasks. They should be able to identify the following:

1. Using log properties to get a single log expression on each side and then use the equality property of logarithms to consider only the arguments of the logs.
2. Using the definition of a logarithm.
3. Taking the log of both sides of an equation.
4. Using the change of base formula.

Tell students that the key idea of this task is to be sure they can use what they know to identify the logical next step needed to solve log and exponential equations. Give students some time to work on their own before working with a partner or group.

**Explore (Small Group):** Monitor students as they work, supporting their thinking on equations that may be unfamiliar. Encourage students to reason about the next possible steps, remembering some of the strategies that were discussed in the launch. Watch for students who have created productive categories for the equations, even if they have not fully developed procedures for that category.

**Discuss (Whole Class):** Begin the discussion by asking students to present solutions to the following problems: 3, 6, 8, 12, 16. As each solution is presented, ask the class to name the steps and record the property or definition that justifies the step.

Then, ask a group to share their classifications. One possible classification system would be:

- Equations with a variable exponent on one side and a number on the other (Problem 3, 10, 19 after being simplified slightly)
  - Equations with exponents on both sides (Problem 12)
  - Equations with a single log expression on each side (Problems 6, 9, 14)
  - Equations with more than one log expression on each side (Problems 13, 16, 18)
  - Equations with a log on one side and a number on the other (Problems 1, 2, 4, 7, 11, 15, 17)
- (Students may break this into sub-categories based upon where the variable is located.)

After sharing one classification system, work with the class to complete and record on their organizers the first few critical steps in solving equation type of equation. (You may need extra copies of the organizer for students that want to make major revisions on what they did during the task.) Before the end of the discussion, be sure to have students share their work on either #16 or #18. These equations end up with two possible solutions, one of which is extraneous. Introduce students to this term and highlight the importance of checking solutions to log equations to be sure that they are in the domain of the log. The only possible solution to problem #14 is extraneous, thus there is no solution.

**Aligned Ready, Set, Go: *Logarithmic Functions 2.9H***

READY, SET, GO!

Name \_\_\_\_\_

Period \_\_\_\_\_

Date \_\_\_\_\_

## READY

Topic: I Comparing solutions and x-intercepts

**Solve for x in each equation.**

1.  $x^2 + x - 2 = 0$

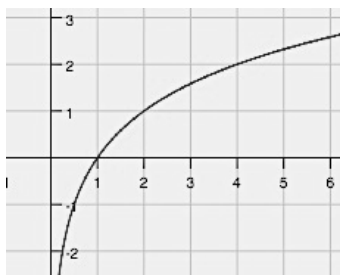
2.  $2x - 6 = 0$

3.  $0 = \log_2 x$

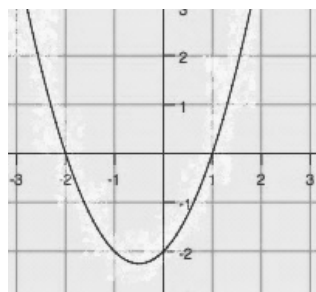
4.  $x^3 - 4x = 0$

5. Match each of the graphs below with one of the above equations. Explain the criteria you used to decide which graph was related to each equation.

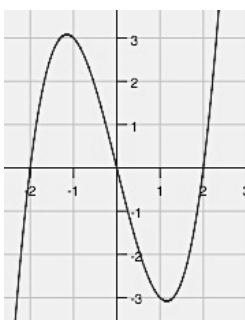
a) Equation?      How did you know?



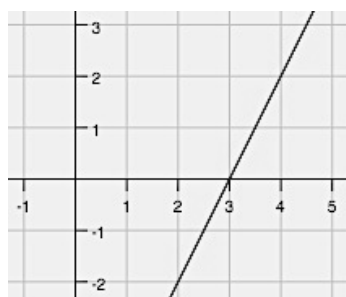
b) Equation?      How did you know?



c) Equation?      How did you know?



d) Equation?      How did you know?



6. Compare the solutions to each equation with the x-intercepts in the graphs. Make a statement about your observations.

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**SET**

Topic: Solving logarithmic equations

**Solve each equation.**

7.  $3 \log_2(x + 4) = 9$

8.  $\log_3(x^2 + 2) = 3$

9.  $\log_5(x^2 + x + 5) = 2$

10.  $2 \log_5 x = 3 \log_5 4$

11.  $\log_7(3x) = \log_7(5x - 8)$

12.  $\log(7x - 12) = 2 \log x$

13.  $4^{1-2x} = 2$

14.  $9^{2x} = 27^{3x-4}$

15.  $5^{x-3} = \sqrt{5}$

16.  $10^{2x} = 80$

17.  $\log_2 4x - \log_2(x - 5) = \log_2 8$

18.  $-3 \log_5 \left( \frac{7x}{10} \right) = 3$

19. On the Richter scale, the magnitude  $R$  of an earthquake of intensity  $I$  is  $R = \log_{10} \frac{I}{I_0}$ , where  $I_0 = 1$  is the minimum intensity used for comparison. If we substitute 1 for  $I_0$ , the formula for measuring the intensity per unit area for an earthquake becomes  $R = \log_{10} I$ .

Find the intensity  $I$  for each of the following earthquakes.

a) San Francisco, California 1906:  $R = 7.7$

b) The World Series Quake, San Francisco, California 1989:  $R = 6.9$

c) The 1906 quake was “only” 0.8 units more on the Richter scale. That doesn’t seem like much. Create a ratio of the intensity of the 1906 quake to the 1989 quake. How many times greater in intensity was the 1906 quake?

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GO

Topic: Reviewing logarithmic functions

Arrange the following expressions in numerical order from smallest to largest. Do not use a calculator. Be prepared to explain your logic.

|     | 1           | 2            | 3           | 4                     | 5            |
|-----|-------------|--------------|-------------|-----------------------|--------------|
| 20. | $\log_3 81$ | $\log_5 125$ | $\log_8 8$  | $\log_4 1$            | $\log 100$   |
| 21. | $\log 200$  | $\log 0.02$  | $\log_2 10$ | $\log_2 \frac{1}{10}$ | $\log_2 200$ |

22. Convert to Logarithmic Form.    a)  $3^6 = 729$     b)  $5^{-2} = 0.04$     c)  $\left(\frac{4}{7}\right)^{-1} = \frac{7}{4}$

23. Convert to Exponential Form.    a)  $\log_6 216 = 3$     b)  $\log_9 1 = 0$     c)  $\log_2 0.5 = -1$

Use the properties of logarithms to rewrite each expression in expanded form. Assume all variables represent positive real numbers.

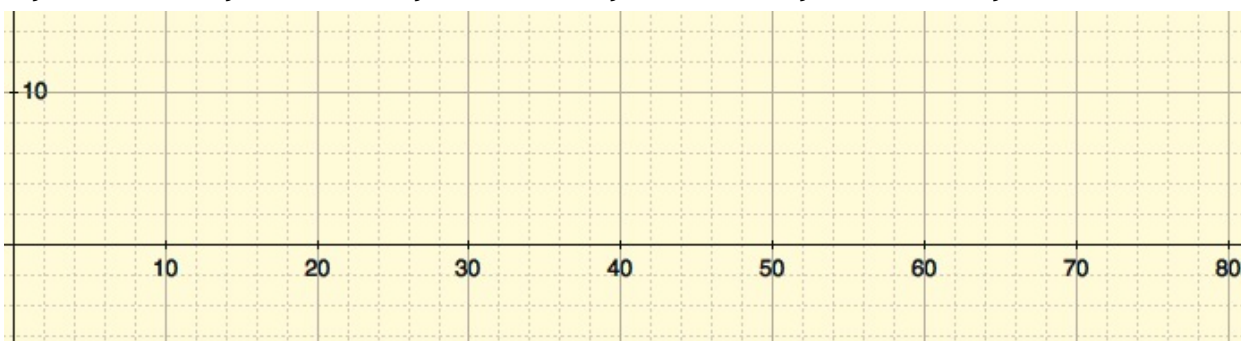
24.  $\log_4 4x^3$

25.  $\log_5 \sqrt{\frac{m}{n}}$

26.  $\log_3 \frac{9w}{xyz}$

27. The number of fish in an aquarium is given by  $f(t) = 4\log(5t + 10)$ , where  $t$  is time in months. Find the number of fish present given the following times. Then graph  $f(t)$ .

a)  $t = 0$     b)  $t = 12$     c)  $t = 24$     d)  $t = 36$     e)  $t = 60$     f)  $t = 72$



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