mathematics vision project

Transforming Mathematics Education

ALGEBRA II

An Integrated Approach

Standard Teacher Notes

MODULE 4

Polynomial Functions

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The Mathematics Vision Project

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POLYNOMIAL FUNCTIONS

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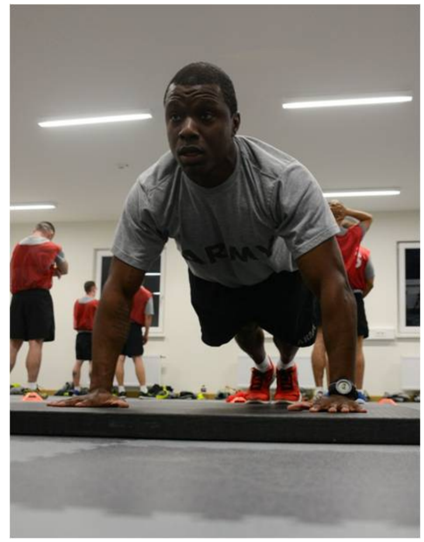

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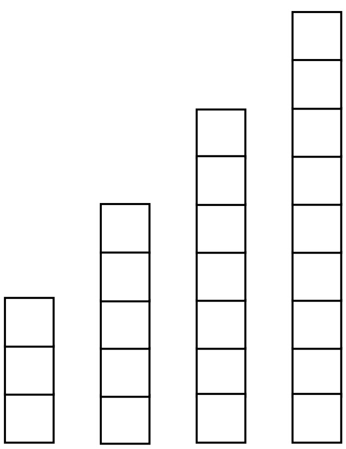
Each year, Scott participates in the "Macho March" promotion. The goal of "Macho March" is to raise money for charity by finding sponsors to donate based on the number of push-ups completed within the month. Last year, Scott was proud of the money he raised, but was also determined to increase the number of push-ups he would complete this year.

Part I: Revisiting the Past

Below is the bar graph and table Scott used last year to keep track of the number of push-ups he completed each day, showing he completed three push-ups on day one and five push-ups (for a combined **total** of eight push-ups) on day two. Scott continued this pattern throughout the month.



n Days	$f(n)$ Push-ups each **day**	$g(n)$ **Total** number of pushups in the month
1	3	3
2	5	8
3	7	15
4	9	24
5	11	35
…	…	
n		

1. Write the recursive and explicit equations for the number of push-ups Scott completed **on any given day** last year. Explain how your equations connect to the bar graph and the table above.

2. Write the recursive and explicit equation for the **accumulated total number of push-ups Scott completed by any given day** during the "Macho March" promotion last year.

Part II: March Madness

This year, Scott's plan is to look at the total number of push-ups he *completed for the month* last year ($g(n)$) and do that many push-ups each day ($m(n)$).

n Days	$f(n)$ Push-ups each **day last year**	$g(n)$ **Total** number of pushups in the month	$m(n)$ Push-ups each **day this year**	$T(n)$ **Total** push-ups completed for the month
1	3	3	3	
2	5	8	8	
3	7	15	15	
4	9	24		
5				
…	…			
n				

3. How many push-ups will Scott complete on day four? How did you come up with this number? Write the recursive equation to represent the total number of push-ups Scott will complete for the month on any given day.

4. How many **total** push-ups will Scott complete for the month on day four?

5. Without finding the explicit equation, make a conjecture as to the type of function that would represent the explicit equation for the total number of push-ups Scott would complete on any given day for this year's promotion.

6. How does the rate of change for this explicit equation compare to the rates of change for the explicit equations in questions 1 and 2?

7. Test your conjecture from question 5 and justify that it will always be true (see if you can move to a generalization for all polynomial functions).

4.1 Scott's March Madness – Teacher Notes

A Develop Understanding Task

Purpose: The purpose of this task is to develop student understanding of how the degree of a polynomial relates to the overall rate of change. Last year, students who did the task Scott's Macho March discovered that quadratic functions can be models for the sum of a linear function, which creates a linear rate of change. This year, students will see that cubic functions can be models for the sum of a quadratic function. Question 7 prompts students to make a conjecture and to move to the generalization that the sum of a polynomial of degree *n* will produce a polynomial of the degree *n+1.* In this task, students have the opportunity to use algebraic, numeric, and graphical representations to model a story context and make connections.

Core Standards Focus

F-BF.1 Write a function that describes a relationship between two quantities.*

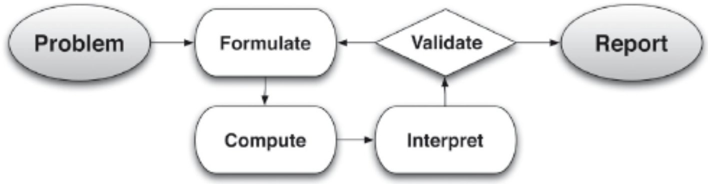
a. Determine an explicit expression, a recursive process, or steps for calculation from a context.
b. Combine standard function types using arithmetic operations. *For example, build a function that models the temperature of a cooling body by adding a constant function to a decaying exponential, and relate these functions to the model.*

F.LE.3 Observe using graphs and tables that a quantity increasing exponentially eventually exceeds a quantity increasing linearly, quadratically, or (more generally) as a polynomial function.

A.CED.2 (Create equations that describe numbers or relationships) Create equations in two or more variables to represent relationships between quantities; graph equations on coordinate axes with labels and scales.

Related Standards: F.IF.9, A.CED.1

This task also follows the structure suggested in the Modeling standard:



Standards for Mathematical Practice:

SMP 1 – Make sense of problems and persevere in solving them

SMP 2 – Reason abstractly and quantitatively

SMP 7 – Look for and make use of structure

SMP 8 – Look for express regularity in repeated reasoning

The Teaching Cycle:

Launch :

Begin the task by setting up the scenario and clarifying the task starts with the number of push-ups Scott completed last year during the event and then moves into the number of push-ups he intends to complete this year. Questions 1 and 2 from the task may be familiar for students from Secondary I (*Scott's Workout*) and Secondary II (*Scott's Macho March*). You may wish to have students work on question one as a warm-up to discuss prior to doing the rest of the task.

Explore :

Have students work independently for a while before they work with a partner. Monitor student thinking as they work. Since students should be familiar with representations for linear functions, do not allow any group to spend too much time on question 1 (you may even wish to make this a 'warm-up' question instead of part of the task). Question 2 is also review but is more difficult. Letting students work through this will be valuable toward making connections and answering the questions in Part II. For those students who have a hard time getting started, encourage them to use another representation (visual model, table, or graph) to help write the equations (both explicit and recursive) as they work on the task. Question 2 answers: Recursive: $g(1) = 3, g(n) = g(n-1) + (2n+1)$; Explicit equation (when simplified): $g(n) = n(n+2)$. The whole group discussion will focus on the questions from Part

II, so look for those who make conjectures that match the purpose of the task. It will be helpful to have students who share to use the visual model they created to explain how they found solutions and made conjectures. For example, a table showing the difference column similar to the one in the answer key (solution 3) to show that cubic functions can be models for the sum of a quadratic function-- just as quadratic functions can be models for the sum of a linear equation.

Discuss :

Start the whole group discussion by having a group go through their process for answering questions 2, 3 and 4 (be sure to have them show the model they used- visual diagram, table, or graph). If possible, have another group share their visual model if they used a different representation (a table highlights the difference column whereas a model or a graph visually highlights the rates of change of the explicit equations).

After students share their strategies for completing questions 2- 4, move the discussion to the last three questions in the task (questions 5-7). These questions focus on the total number of push-ups completed for the month, or in other words, the *cubic rate of change* created as a result of adding the explicit quadratic function to current number of push-ups completed on any given day. The goal for this portion of the discussion is for students to summarize the key points that reflect the purpose of this task*.

Observations to be made:

- The first difference is quadratic, the second difference is linear and the third difference is constant (a characteristic of a cubic function).
- The graph "curves" (grows at a faster rate) more than that of a quadratic (This also helps for future work regarding rates of change of polynomial functions versus that of exponential functions: while cubic functions grow at a faster rate than quadratics, they are not as fast as exponential, whose third difference is still exponential).
- Cubic functions can be models for the sum of the terms of a quadratic function in the same way that quadratic functions can be models for sum of terms of a linear function. (This is similar to the relationship from the Secondary II task *Scott's Macho March* where students discovered that quadratic functions can be models for the sum of a linear function, which creates a linear rate of change.)

- Recognize there is a relationship of the degree of a polynomial and the overall rate of change. (Generalization: when the *nth* difference is constant, then the polynomial is of degree n).
- Make a conjecture that the sum of a polynomial of degree n will produce a polynomial of the degree $n+1$.

*If students struggle to come up with the above observations, have them focus on the difference columns and compare the three recursive equations from questions 1, 2, and 3 (whose equations represent the former value plus a constant term, a linear expression, and a quadratic expression respectively). Ask students to identify similarities and differences. The important thing to notice is that *the change in the function is the expression added to the recursive formula*. If the rate of change is a constant, then the function is linear. If the change is a linear expression, then the function is quadratic. If the change is a quadratic expression, then the function is cubic.

**If time permits, it is always good to review and discuss features of functions. In this task, the function to describe the total number of push-ups Scott would complete would be a good discussion for domain/range, where the situation is increasing/decreasing, intercepts, whether the function is discrete or continuous, and the 'end behavior'. While the domain is restricted to the number of days in March, this is a good example of distinguishing between the range of the situation versus the end behavior of the actual function.

Aligned Ready, Set, Go: *Polynomial Functions 4.1*

READY, SET, GO! Name Period Date

READY

Topic: Completing inequality statements

For each problem, place the appropriate inequality symbol between the two expressions to make the statement true.

$If\ a > b, then:$	$If\ x > 10, then:$	$If\ 0 < x < 1$
1. $3a$ ___ $3b$	4. x^2 ___ 2^x	7. x ___ x^2
2. $b - a$ ___ $a - b$	5. $\sqrt{x}$ ___ x^2	8. $\sqrt{x}$ ___ x
3. $a + x$ ___ $b + x$	6. x^2 ___ x^3	9. x ___ $3x$

SET

Topic: Classifying functions

Identify the type of function for each problem. Explain how you know.

10

x	$f(x)$
1	3
2	6
3	12
4	24
5	48

11.

x	$f(x)$
1	3
2	6
3	9
4	12
5	15

12.

x	$f(x)$
1	3
2	9
3	18
4	30
5	45

13.

x	$f(x)$
1	7
2	9
3	13
4	21
5	37

14.

x	$f(x)$
1	-26
2	-19
3	0
4	37
5	98

15.

x	$f(x)$
1	-4
2	3
3	18
4	41
5	72

16. Which of the above functions are NOT polynomials?

GO

Topic: Recalling long division and the meaning of a factor

Find the quotient without using a calculator. If you have a remainder, write the remainder as a whole number. Example: $21\overset{\displaystyle 7}{\overline{\smash{)}149}}$ **remainder 2**

17. $30\overline{\smash{)}510}$

18. $13\overline{\smash{)}8359}$

19. Is 30 a factor of 510? How do you know?

20. Is 13 a factor of 8359? How do you know?

21. $22\overline{\smash{)}14857}$

22. $952\overline{\smash{)}40936}$

23. Is 22 a factor of 14587? How do you know?

24. Is 952 a factor of 40936? How do you know?

25. $92\overline{\smash{)}3405}$

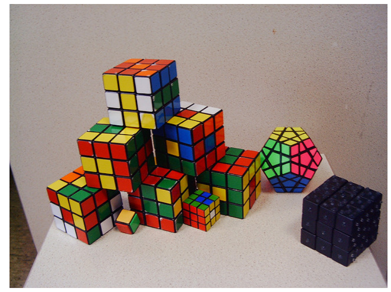
26. $27\overline{\smash{)}3564}$

27. Is 92 a factor of 3405?

28. Is 27 a factor of 3564?

4.2 You-mix Cubes

A Solidify Understanding Task

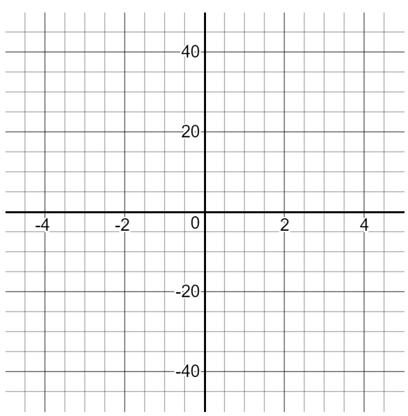


In Scott's March Madness, the function that was generated by the sum of terms in a quadratic function was called a **cubic function**. Linear functions, quadratic functions, and cubic functions are all in the family of functions called **polynomials**, which include functions of higher powers too. In this task, we will explore more about cubic functions to help us to see some of the similarities and differences between cubic functions and quadratic functions.

To begin, let's take a look at the most basic cubic function, $f(x) = x^3$. It is technically a **degree 3 polynomial** because the highest exponent is 3, but it's called a cubic function because these functions are often used to model volume. This is like quadratic functions which are **degree 2** polynomials but are called quadratic after the Latin word for square. Scott's March Madness showed that linear functions have a constant rate of change, quadratic functions have a linear rate of change, and cubic functions have a quadratic rate of change.

1. Use a table to verify that $f(x) = x^3$ has a quadratic rate of change.

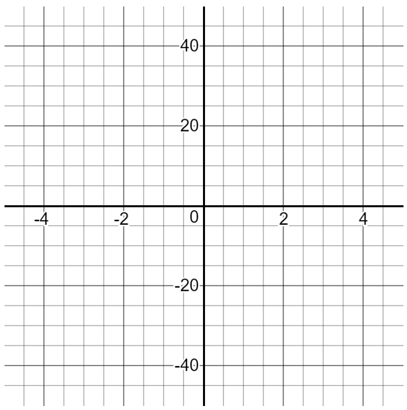
2. Graph $f(x) = x^3$.



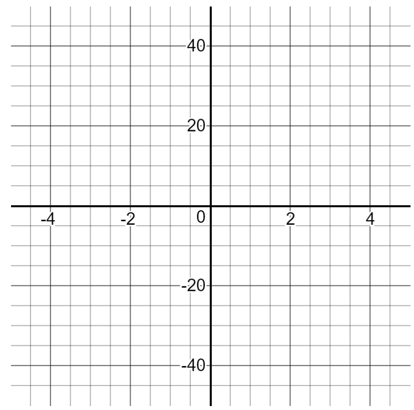
3. Describe the features of $f(x) = x^3$ including intercepts, intervals of increase or decrease, domain, range, etc.

4. Using your knowledge of transformations, graph each of the following without using technology.

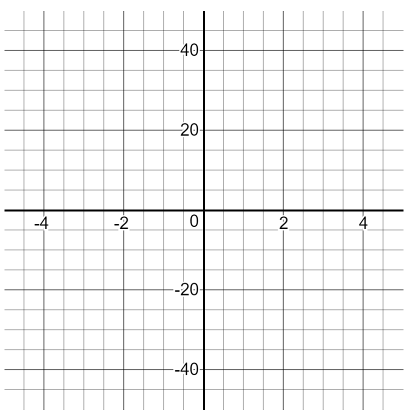
a) $f(x) = x^3 - 3$



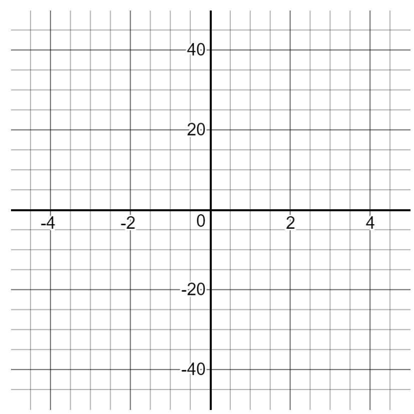
b) $f(x) = (x + 3)^3$



c) $f(x) = 2x^3$



d) $f(x) = -(x - 1)^3 + 2$



5. Use technology to check your graphs above. What transformations did you get right? What areas do you need to improve on so that your cubic graphs are perfect?

6. Since quadratic functions and cubic functions are both in the polynomial family of functions, we would expect them to share some common characteristics. List all the similarities between $f(x) = x^3$ and $g(x) = x^2$.

7. As you can see from the graph of $f(x) = x^3$, there are also some real differences in cubic functions and quadratic functions. Each of the following statements describe one of those differences. Explain why each statement is true by completing the sentence.

a) The range of $f(x) = x^3$ is $(-\infty, \infty)$, but the range of $g(x) = x^2$ is $[0, \infty)$ because:

____

b) For $x > 1, f(x) > g(x)$ because: ______________________________

____

c) For $0 < x < 1, g(x) > f(x)$ because: ______________________________

____

4.2 You-mix Cubes – Teacher Notes

A Solidify Understanding Task

Purpose:

The purpose of this task is to examine and extend ideas about cubic functions that were surfaced in *3.1 Scott's Macho March Madness*. Students consider the basic cubic function, $f(x) = x^3$, identifying its characteristics and graph. Students will recognize that the graph of $f(x) = x^3$ can be transformed using the same techniques as quadratic functions. Students will also make other comparisons between cubic and quadratic functions, specifically, their end behavior and the values of each function when x is a fraction.

Core Standards Focus:

F.BF.3 Identify the effect on the graph of replacing *f(x) by f(x)+k, kf(x), f(kx), and f(x+k)* for specific values of k (both positive and negative); find the value of *k* given the graphs. Experiment with cases and illustrate an explanation of the effects on the graph using technology. Include recognizing even and odd functions from the graphs and algebraic expressions for them.

F.IF.4. For a function that models a relationship between two quantities, interpret key features of graphs and tables in terms of the quantities, and sketch graphs showing key features given a verbal description of the relationship. Key features include: intercepts; intervals where the function is increasing, decreasing, positive, or negative; relative maximums and minimums; symmetries; end behavior; and periodicity.

F.IF.5. Relate the domain of a function to its graph and, where applicable, to the quantitative relationship it describes. For example, if the function *h(n)* gives the number of person-hours it takes to assemble *n* engines in a factory, then the positive integers would be an appropriate domain for the function.

F.IF.7: Graph functions expressed symbolically and show key features of the graph, by hand in simple cases and using technology for more complicated cases.

a. Graph linear and quadratic functions and show intercepts, maxima, and minima.

c. Graph polynomial functions, identifying zeros when suitable factorizations are available, and showing end behavior.

Related Standards: A.SSE.1

Standards for Mathematical Practice:

SMP 2 – Reason abstractly and quantitatively

SMP 8 – Look for express regularity in repeated reasoning

Vocabulary: cubic function, polynomial function, degree of a polynomial

The Teaching Cycle:

Launch:

Begin the task by telling students that the cubic functions introduced in task *3.1 Scott's Macho March Madness* and the quadratic and linear functions that they have previously worked with all fall into the family of polynomial functions. A technical definition of polynomial is given later, but a working definition is that a polynomial function is a sum of terms in the same variable with the natural number exponents and real coefficients. Demonstrate for students how to identify the degree of the polynomial based on the highest-powered term:

Degree 1 Polynomial (Linear Function): $y = 3x - 1$

Degree 2 Polynomial (Quadratic Function): $y = 3x^2 - 2x + 9$

Degree 3 Polynomial (Cubic Function): $y = -5x^3 + 3x^2 - 10x + 6$

Degree 4 Polynomial (Quartic Function): $y = x^4 + 7x^3 - 4x^2 - 24x + 1$

Explain that in this task, we will be introducing more of the characteristics of cubic functions and comparing them to what we know about quadratic functions.

Explore:

Monitor students as they are working on the task to see that they are creating an organized table of values that can be used throughout the task. In the table for problem #1, it is easiest to show the quadratic rate of change starting from 0 and looking at whole number values. If students are stuck, encourage them to start there. Eventually, they will need to consider both positive and negative numbers for the graph, which can be easily added to the table. Listen for students who are expressing the idea that the values on the left side of the graph are just the opposite of the values on the right side of the graph.

As students are working on the transformations, support them in connecting to their previous work with functions and identifying the transformations first, and then using anchor points from the original graph to find their new graph.

Discuss:

Begin the discussion by having a student present a table of values that shows the first, second, and third differences. Ask the class to explain how the differences relate to each other, i.e. a constant third difference means that the second difference is linear, a linear second difference means that the first difference is quadratic. Remind students that, generally, functions are defined by their rates of change, and the patterns in rates of change that we anticipated in *3.1 Scott's Macho March Madness* are consistent across polynomials.

Move the discussion to the graph in problem #2. Ask a student to share how they used the table to create the graph and how they can see the rate of change in the graph. Ask students to describe why the values to the left of zero are negative for a cubic function and reinforce the idea that when negative numbers are squared, the product is positive and when negative numbers are cubed, the product is negative. Tell students that their tables will help them with the anchor points they need to transform the graph of $f(x) = x^3$. They should be able to easily know the points (0,0), (1,1), (-1,-1), (2, 8), and (-2, -8) and use them for the transformations.

Continue the discussion by asking students to share their work on the graph in question #4. Be sure that students demonstrate how they determined the transformations and then used the anchor points to locate and graph the new function.

Finally, discuss the remaining questions. Question 7c may produce a lively discussion since students often believe that cubing a number always makes it bigger than squaring the number. After discussing how this belief is not true for fractions, project a graph of $f(x) = x^3$ and $g(x) = x^2$ together and zoom in on each of the following intervals to see which function is greater in the interval: $(-\infty, -1)$, $(-1, 0)$, $(0, 1)$, $(1, \infty)$.

Aligned Ready, Set, Go: ***Polynomial Functions 4.2***

READY, SET, GO! Name Period Date

READY

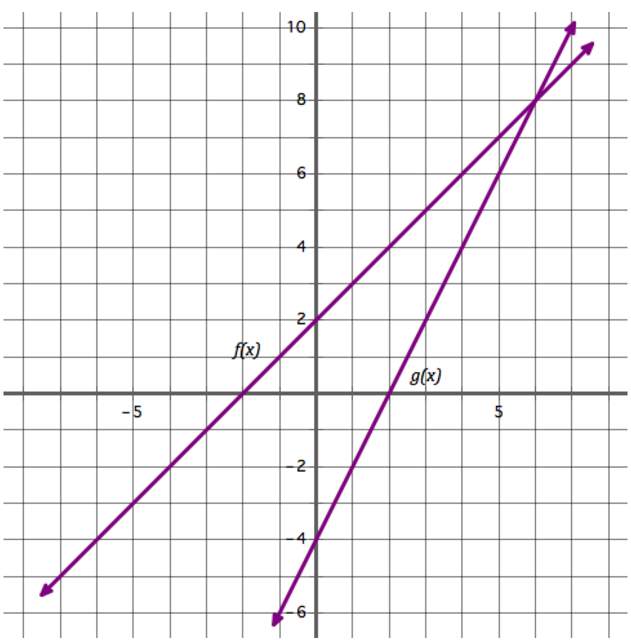
Topic: Adding and subtracting binomials

Add or subtract as indicated.

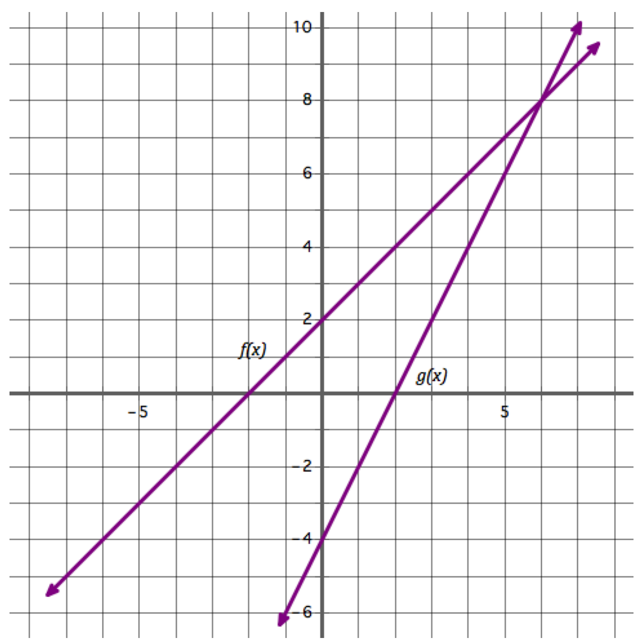
1. $(6x + 3) + (4x + 5)$
2. $(x + 17) + (9x - 13)$
3. $(7x - 8) + (-2x + 9)$
4. $(4x + 9) - (x + 2)$
5. $(-3x - 1) - (2x + 5)$
6. $(8x + 3) - (-10x - 9)$
7. $(3x - 7) + (-3x - 7)$
8. $(-5x + 8) - (-5x + 7)$
9. $(8x + 9) - (7x + 9)$

10. Use the graphs of $f(x)$ and $g(x)$ to sketch the graphs of $f(x) + g(x)$ and $f(x) - g(x)$.

$f(x) + g(x)$



$f(x) - g(x)$.



SET

Topic: Comparing simple polynomials

11. Complete the tables below for $y = x$ and $y = x^3$ and $y = x^5$

x	$y = x$
---−1	
0	
1	

x	$y = x^3$
−1	
0	
1	

x	$y = x^5$
−1	
0	
1	

12. What assumption might you be tempted to make about the graphs of $y = x$, $y = x^3$ and $y = x^5$ based on the values you found in the 3 tables above?

13. What do you really know about the graphs of $y = x$ and $y = x^3$ and $y = x^5$ despite the values you found in the 3 tables above?

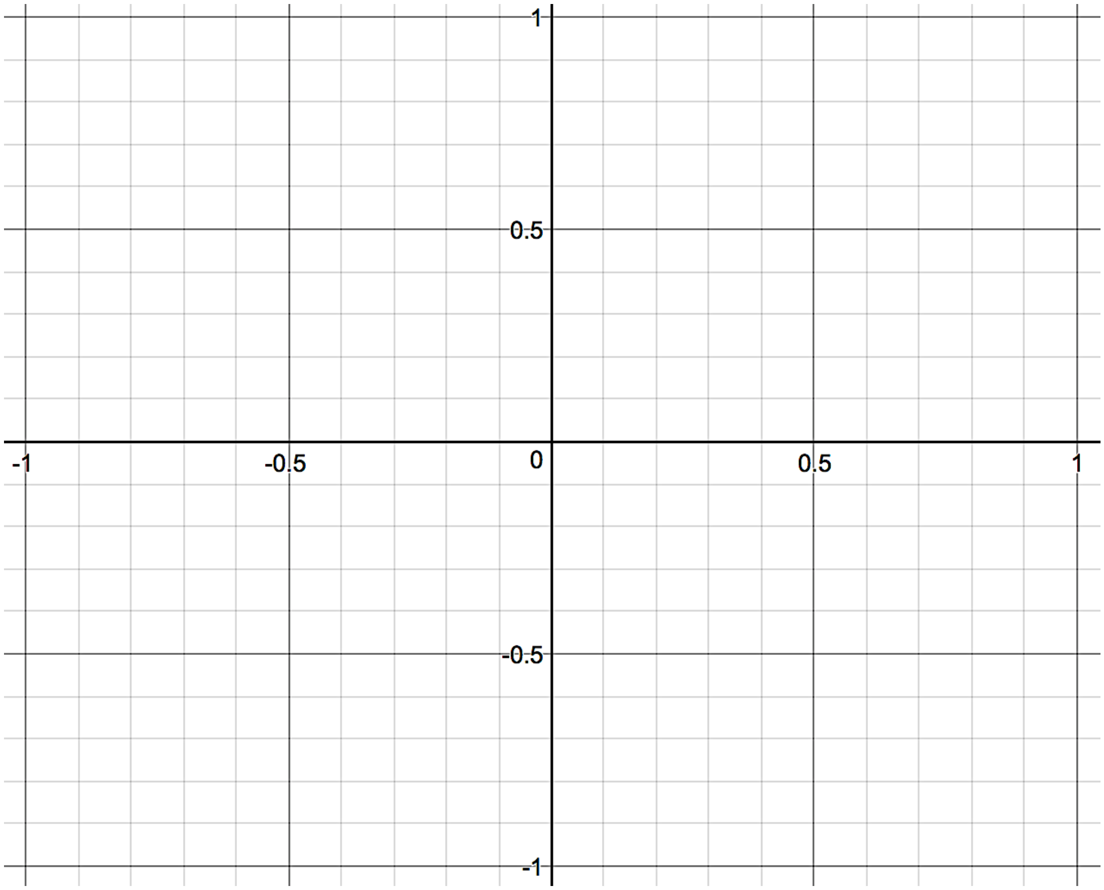
14. Complete the tables with the additional values.

x	$y = x$
-1	
$-{}^{1}/_{2}$	
0	
${}^{1}/_{2}$	
1	

x	$y = x^3$
-1	
$-{}^{1}/_{2}$	
0	
${}^{1}/_{2}$	
1	

x	$y = x^5$
-1	
$-{}^{1}/_{2}$	
0	
${}^{1}/_{2}$	
1	

15. Graph $y = x$ *and* $y = x^3$ *and* $y = x^5$ on the interval $[-1, 1]$, using the same set of axes.



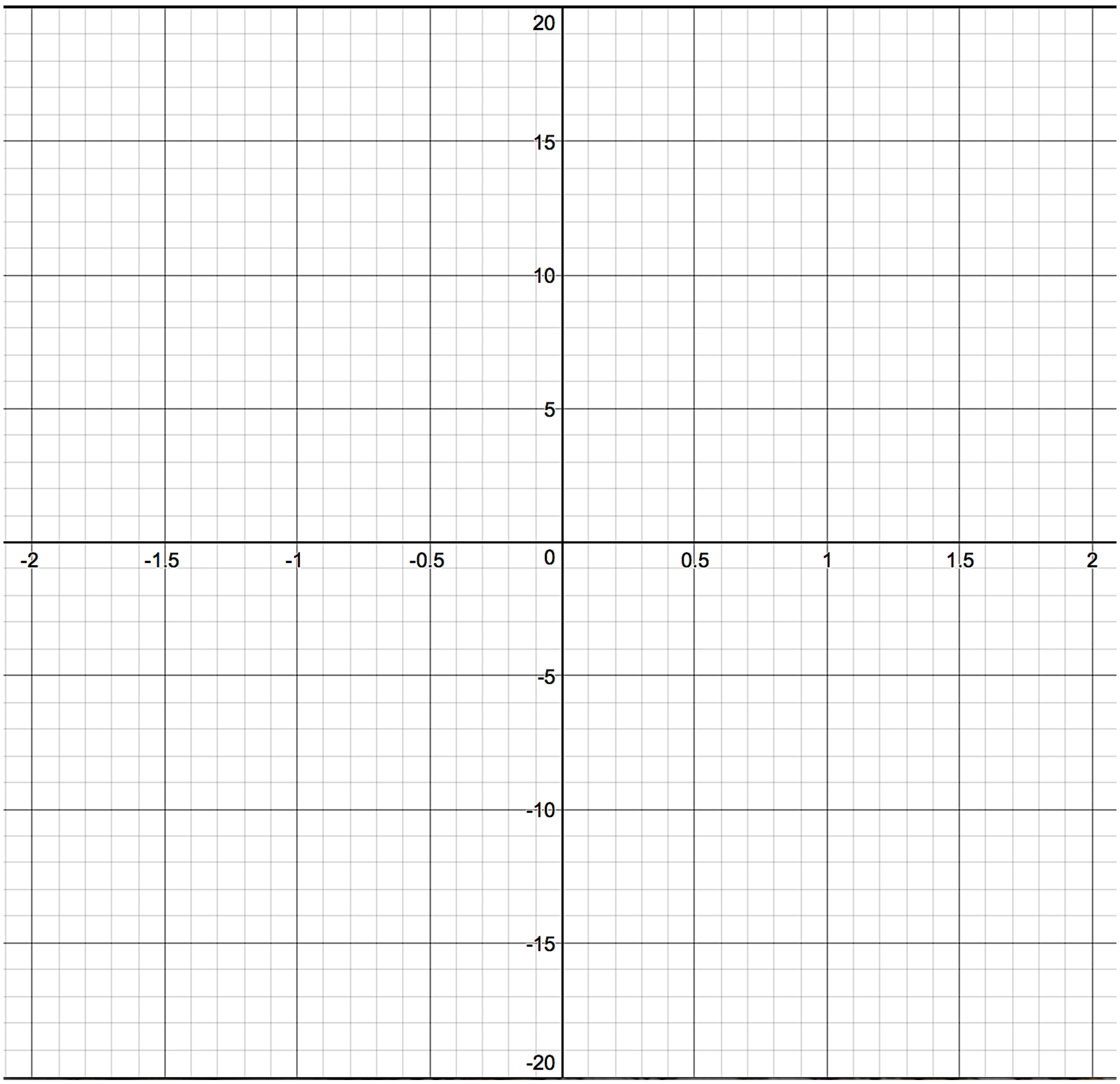
16. Complete the tables with the additional values.

x	$y = x$
-2	
-1	
$-1/2$	
0	
$1/2$	
1	
2	

x	$y = x^3$
-2	
-1	
$-1/2$	
0	
$1/2$	
1	
2	

x	$y = x^5$
-2	
-1	
$-1/2$	
0	
$1/2$	
1	
2	

17. Graph $y = x \text{ and } y = x^3 \text{ and } y = x^5$ on the interval [-2, 2], using the same set of axes.



GO

Topic: Using the exponent rules to simplify expressions

Simplify.

18. $x^{1/3} \cdot x^{1/6} \cdot x^{1/4}$

19. $a^{2/5} \cdot a^{3/10} \cdot a^{2/15}$

20. $m^{4/7} \cdot m^{3/14} \cdot m^{5/28}$

4.3 Building Strong Roots

A Solidify Understanding Task

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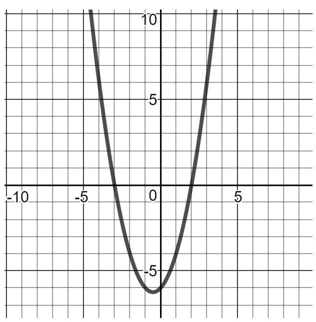
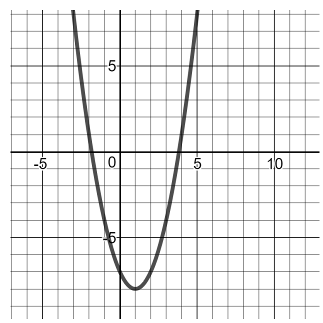
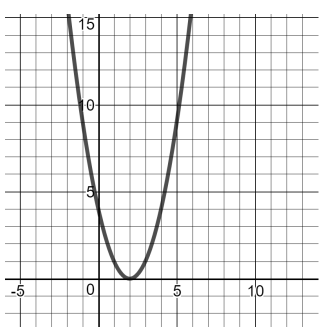
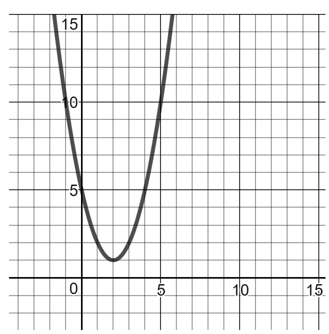
When working with quadratic functions, we learned the Fundamental Theorem of Algebra:
An n^{th} degree polynomial function has n roots.
In this task, we will be exploring this idea further with other polynomial functions.

First, let's brush up on what we learned about quadratics. The equations and graphs of four different quadratic equations are given below. Find the roots for each and identify whether the roots are real or imaginary.

1.

a) $f(x) = x^2 + x - 6$ 	b) $g(x) = x^2 - 2x - 7$ 
Roots:	Roots:
Type of roots:	Type of roots:
c) $h(x) = x^2 - 4x + 4$ 	d) $k(x) = x^2 - 4x + 5$ 
Roots:	Roots:
Type of roots:	Type of roots:

2. Did all of the quadratic functions have 2 roots, as predicted by the Fundamental Theorem of Algebra? Explain.

3. It's always important to keep what you've previously learned in your mathematical bag of tricks so that you can pull it out when you need it. What strategies did you use to find the roots of the quadratic equations?

4. Using your work from problem 1, write each of the quadratic equations in factored form. When you finish, check your answers by graphing, when possible, and make any corrections necessary.

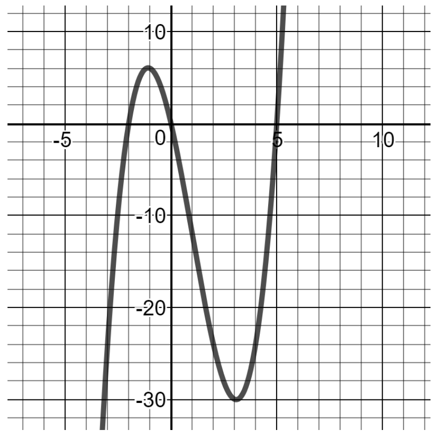
a) $f(x) = x^2 + x - 6$	b) $g(x) = x^2 - 2x - 7$
Factored form:	Factored form:
c) $h(x) = x^2 - 4x + 4$	d) $k(x) = x^2 - 4x + 5$
Factored form:	Factored form:

5. Based on your work in problem 1, would you say that roots are the same as x-intercepts? Explain.

6. Based on your work in problem 4, what is the relationship between roots and factors?

Now let's take a closer look at cubic functions. We've worked with transformations of $f(x) = x^3$, but what we've seen so far is just the tip of the iceberg. For instance, consider:

$g(x) = x^3 - 3x^2 - 10x$



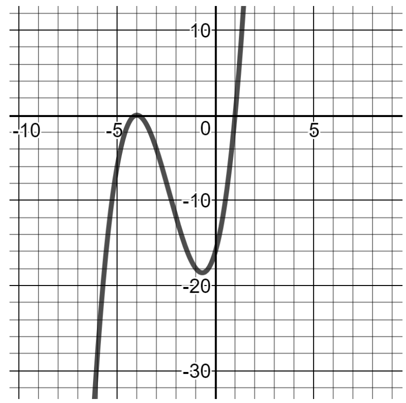
7. Use the graph to find the roots of the cubic function. Use the equation to verify that you are correct. Show how you have verified each root.

8. Write $g(x)$ in factored form. Verify that the factored form is equivalent to the standard form.

9. Are the results you found in #7 consistent with the Fundamental Theorem of Algebra? Explain.

Here's another example of a cubic function.

$f(x) = x^3 + 7x^2 + 8x - 16$

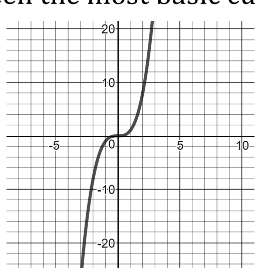


10. Use the graph to find the roots of the cubic function.

11. Write $f(x)$ in factored form. Verify that the factored form is equivalent to the standard form. Make any corrections needed.

12. Are the results you found in #10 consistent with the Fundamental Theorem of Algebra? Explain.

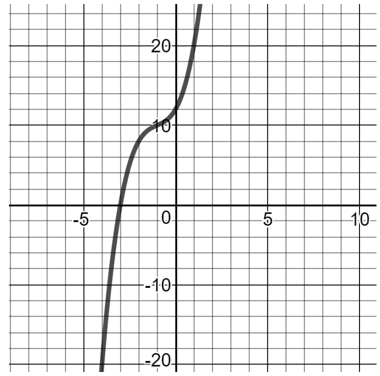
13. We've seen the most basic cubic polynomial function, $h(x) = x^3$ and we know its graph looks like this:



Explain how $h(x) = x^3$ is consistent with the Fundamental Theorem of Algebra.

14. Here is one more cubic polynomial function for your consideration. You will notice that it is given to you in factored form. Use the equation and the graph to find the roots of $p(x)$.

$p(x) = (x+3)(x^2+4)$



15. Use the equation to verify each root. Show your work below.

16. Are the results you found in #14 consistent with the Fundamental Theorem of Algebra? Explain.

17. Explain how to find the factored form of a polynomial, given the roots.

18. Explain how to find the roots of a polynomial, given the factored form.

4.3 Building Strong Roots – Teacher Notes
A Solidify Understanding Task

Purpose: The purpose of this task is to extend the Fundamental Theorem of Algebra from quadratic functions to cubic functions. The task asks students to use graphs and equations to find roots and factors and to consider the relationship between them. Students will also consider quadratic and cubic functions with multiple real roots and imaginary roots.

Core Standards Focus:

A.SSE.1 Interpret expressions that represent a quantity in terms of its context.

a. Interpret parts of an expression, such as terms, factors, and coefficients.

A.APR.3 Identify zeros of polynomials when suitable factorizations are available, and use the zeros to construct a rough graph of the function defined by the polynomial.

N.CN.9 Know the Fundamental Theorem of Algebra; show that it is true for quadratic polynomials. (For Secondary III, limit to polynomials with real coefficients).

Standards for Mathematical Practice:

SMP 7 – Look for and make use of structure

Vocabulary: roots, factors, x-intercepts, multiplicity

The Teaching Cycle:

Launch (Whole Group):

Begin the task by introducing the Fundamental Theorem of Algebra. Students were introduced to this theorem in relation to quadratic functions in previous courses, so the first part of this task is to activate background knowledge so that their understanding and strategies for quadratic functions can be extended to cubic functions. Tell students that today's work will involve determining the number of roots for functions as well as the relationship between roots and factors. Students may need to be reminded that roots are the values of x that make $f(x) = 0$. Then, ask students to work on questions 1-6.

Explore (Small Group):

As students are working, support them in using the graphs to find roots. Ask questions to help them to make connections between the equations and the graphs. Listen for students who are using quadratic formula to find the exact roots when they are not obvious on the graph. Some students may try to estimate the roots; nudge them to think about what they could do to find exact values. In the case where the roots are not evident from the graph, ask students how they could use the equation to find roots. When students are working on finding factors from roots, you may need to remind them of how they solved equations in factored form, so that they think about how to work backwards.

Discuss (Whole Group):

Begin the discussion with the function in #1b. Ask a previously-selected student to share their work in finding the exact values of the roots using quadratic formula. Ask students to find the approximate values and ask where those values can be seen on the graph. Ask another student to show how they wrote the factors for the roots in #1b.

Next, ask a student to share how they found the roots for #1c and wrote it in factored form (question 2c). Ask the class if this function has the number of roots predicted by the Fundamental Theorem of Algebra. This may generate some controversy. After allowing students to share arguments, tell them that the factorization helps us to see that this function has a multiple root. Explain that it is said that this is a "root with multiplicity of 2". Remind students that for quadratics with multiple roots, the vertex is generally on the x-axis.

Have a student share their work finding the roots and factors for question 2d. This will serve as a brief reminder of imaginary numbers, which will be used in this task and upcoming tasks.

Discuss questions 5 and 6. Many students may argue that roots and x-intercepts are the same thing. If so, ask students if the roots were x-intercepts in question 2d. Explain that a root is defined as the value of x that makes $f(x) = 0$. An x-intercept is defined to be the point where a graph crosses the x-axis. If a function has imaginary roots, it will not cross the x-axis at the roots, so the roots are not x-intercepts. Conventionally, a root is written as $x = a$, and x-intercepts are written as points, e.g. $(a, 0)$. Remind students of the relationship between roots and factors, then ask them to work on the remainder of the task.

Explore (Small Groups):
Support students as they are working to apply some of the techniques that they learned with quadratic functions to cubic functions. They should start with looking for roots that can be easily identified from the graph or a factor, and then use the remaining factor to find the roots. As students are working on questions 10 and 11, look for a student that initially thought that $(x + 4)$ is a single root, and then discovered that it must be a multiple root to make the equation cubic. Also look for a student that has found the imaginary roots in #14.

Discuss (Whole Group):
Start the second discussion with questions #10 and #11. Have a student selected during the *Explore* time share his/her work in finding the roots and how they determined that $(x + 4)$ must be a multiple root. Ask the class to compare how the cubic graph looks when there is a multiple root and how the quadratic graph looked with the multiple root. This will help students to recognize that when a polynomial touches the x-axis without crossing through, it probably contains a multiple root of even multiplicity at that point.

Ask students to respond to #13. There may be some disagreement about this question. Again, have students share ideas and come to the conclusion that it must have a multiple root at $x = 0$. Point out that the graph crosses the x-axis at this multiple root, which makes it harder to identify as a multiple root than when the graph touches the axis and bounces off, as it does with an even number of multiple roots.

Ask previously-selected students to share their work on #14 and #15 that demonstrates one method for solving a quadratic to find imaginary roots. Point out that there are two imaginary roots and ask if that would always happen. Students may recognize that using the quadratic formula or square-rooting both sides of the equation would always lead to two imaginary roots in similar cases.

Wrap the discussion up with the last two questions (17 and 18), demonstrating how to use the roots to get factors and how to use factored form of a polynomial to find roots.

Aligned Ready, Set, Go: *Polynomial Functions 4.3*

READY, SET, GO! Name Period Date

READY

Topic: Practicing long division on polynomials

Divide using long division. (These problems have no remainders. If you get one, try again.)

1. $(x+3)\overline{)5x^3+2x^2-45x-18}$

2. $(x-6)\overline{)x^3-x^2-44x+84}$

3. $(x-5)\overline{)3x^3-15x^2+12x-60}$

4. $(x+2)\overline{)x^4+6x^3+7x^2-6x-8}$

SET

Topic: Applying the Fundamental Theorem of Algebra

Predict the number of roots for each of the given polynomial equations. (Remember that the Fundamental Theorem of Algebra states: An n^{th} degree polynomial function has n roots.)

5. $a(x) = x^2 + 3x - 10$

6. $b(x) = x^3 + x^2 - 9x - 9$

7. $c(x) = -2x - 4$

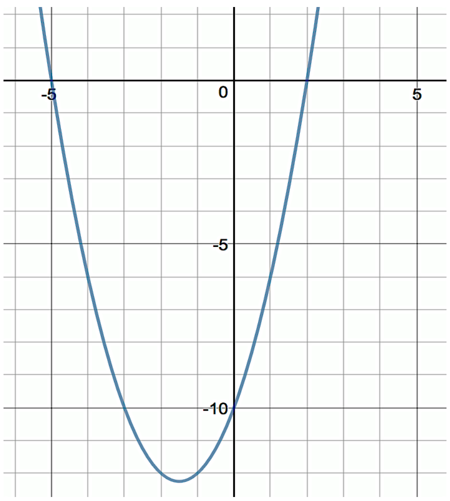
8. $d(x) = x^4 - x^3 - 4x^2 + 4x$

9. $f(x) = -x^2 + 6x - 9$

10. $g(x) = x^6 - 5x^4 + 4x^2$

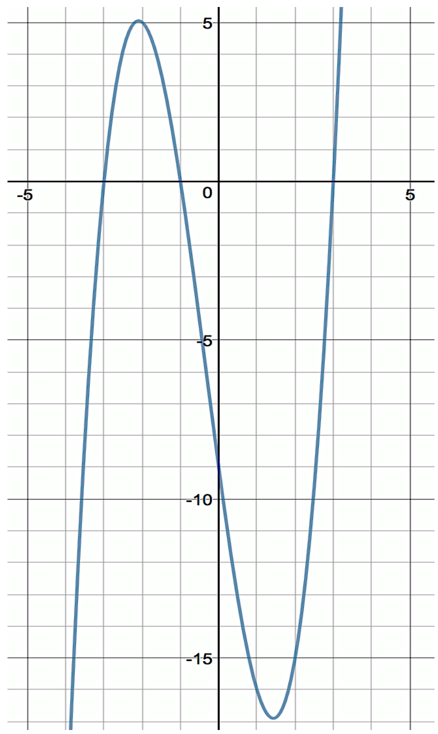
Below are the graphs of the polynomials from the previous page. Check your predictions. Then use the graph to help you write the polynomial in factored form.

11. $a(x) = x^2 + 3x - 10$



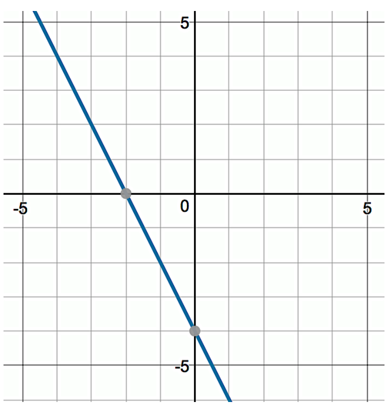
Factored form:

12. $b(x) = x^3 + x^2 - 9x - 9$



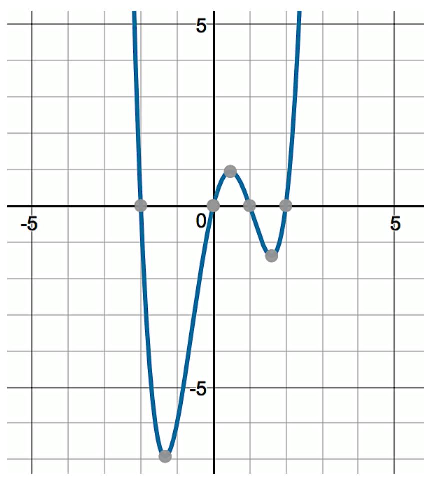
Factored form:

13. $c(x) = -2x - 4$



Factored form:

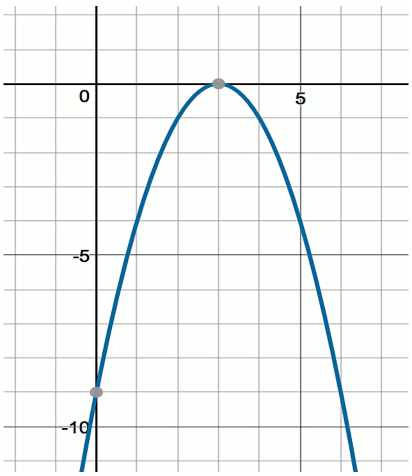
14. $d(x) = x^4 - x^3 - 4x^2 + 4x$



Factored form:

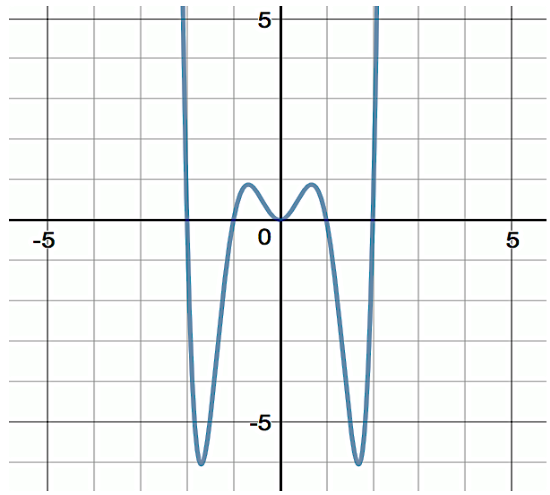
Need help? Visit **www.rsgsupport.org**

15. $f(x) = -x^2 + 6x - 9$



Factored form:

16. $g(x) = x^6 - 5x^4 + 4x^2$



Factored form:

17. The graphs of #15 and #16 don't seem to follow the Fundamental Theorem of Algebra, but there is something similar about each of the graphs. Explain what is happening at the point (3, 0) in #15 and at the point (0,0) in #16.

GO

Topic: Solving quadratic equations

Find the zeros for each equation using the quadratic formula.

18. $f(x) = x^2 + 20x + 51$

19. $f(x) = x^2 + 10x + 25$

20. $f(x) = 3x^2 + 12x$

21. $f(x) = x^2 - 11$

22. $f(x) = x^2 + x - 1$

23. $f(x) = x^2 + 2x + 3$

4.4 Getting to the Root of the Problem

A Solidify Understanding Task

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In *4.3 Building Strong Roots,* we learned to predict the number of roots of a polynomial using the Fundamental Theorem of Algebra and the relationship between roots and factors. In this task, we will be working on how to find all the roots of a polynomial given in standard form.

Let's start by thinking again about numbers and factors.

1. If you know that 7 is a factor of 147, what would you do to find the prime factorization of 147? Explain your answer and show your process here:

2. How is your answer like a polynomial written in the form: $P(x) = (x-7)^2(x-3)$?

The process for finding factors of polynomials is exactly like the process for finding factors of numbers. We start by dividing by a factor we know and keep dividing until we have all the factors. When we get the polynomial broken down to a quadratic, sometimes we can factor it by inspection, and sometimes we can use our other quadratic tools like the quadratic formula.

Let's try it! For each of the following functions, you have been given one factor. Use that factor to find the remaining factors, the roots of the function, and write the function in factored form.

3. Function: $f(x) = x^3 + 3x^2 - 4x - 12$ Factor: $(x+3)$ Roots of function:

Factored form:

4. Function: $f(x) = x^3 + 6x^2 + 11x + 6$ Factor: $(x+1)$ Roots of function:

Factored form:

5. Function: $f(x) = x^3 - 5x^2 - 3x + 15$ Factor: $(x-5)$ Roots of function:

Factored form:

6. Function: $f(x) = x^3 + 3x^2 - 12x - 18$ Factor: $(x-3)$ Roots of function:

Factored form:

7. Function: $f(x) = x^4 - 16$ Factor: $(x-2)$ Roots of function:

Factored form:

8. Function: $f(x) = x^3 - x^2 + 4x - 4$ Factor: $(x - 2i)$ Roots of function:

Factored form:

9. Is it possible for a polynomial with real coefficients to have only one imaginary root? Explain.

10. Based on the Fundamental Theorem of Algebra and the polynomials that you have seen, make a table that shows all the number of roots and the possible combinations of real and imaginary roots for linear, quadratic, cubic, and quartic polynomials.

4.4 Getting To The Root Of The Problem – Teacher Notes

A Solidify Understanding Task

Purpose:
The purpose of this task is for students to find roots of polynomials and write the polynomials in factored form. This task builds on previous algebraic work, including factoring, polynomial long division, and quadratic formula. Students also use their knowledge of the Fundamental Theorem of Algebra to know how many roots a function has and to consider the possible combinations of real and imaginary roots for polynomials of degree 1-4. Students learn that imaginary roots occur in conjugate pairs and use this knowledge to both find roots and know the possible combinations of roots for polynomials.

Core Standards Focus:
A.APR.3: Identify zeros polynomials when suitable factorizations are available, and use the zeros to construct a rough graph of the function defined by the polynomial.

N.CN.8: Extend polynomial identities to the complex numbers. *For example, rewrite* $x^2 + 4\ as\ (x + 2i)(x - 2i)$.

N.CN.9: Know the Fundamental Theorem of Algebra; show that it is true for quadratic polynomials. (In Secondary III, limit to polynomials with real coefficients).

Standards for Mathematical Practice:

SMP 1 – Make sense of problems and persevere in solving them

SMP 8 – Look for express regularity in repeated reasoning

Vocabulary: conjugate pairs

The Teaching Cycle:

Launch:

Begin the task by asking students about question 1, what is the prime factorization of 147 and how did you find it? Students will describe a process of dividing by the known factor and then breaking it down until all the factors are prime. This is analogous to the work that will be done in this task. We'll be given a polynomial and a known factor. We'll divide by the known factor and then break down the result until all the factors are linear. Question 2 illustrates that both polynomials and numbers can be written in factored form or standard form. Tell students that they will need to use everything they know about polynomials and all the tools in their toolkit, like quadratic formula, to break down the polynomials and find their roots and factors.

Explore:

As students begin working on problems #3 and #4, you may encounter some students who have graphed the function and seen that they both have integer roots that are easily identified from the graph. If so, ask them to verify the roots, either by long division or substitution into the function. If this occurs, you may wish to ask one of these students to present their work to the class, since connecting the graph to the roots is a very useful strategy.

Support students as they are working to use factoring, quadratic formula, taking the square root of both sides, etc. to solve the equations and find the roots. Look for students that have selected efficient strategies, rather than just one consistent strategy, to share. Students should be exposed to the idea that the quadratic formula is a powerful tool, but not always the most efficient tool.

Students are likely to have some trouble with #8. They can try to use long division, although it is algebraically more complicated than what they have seen previously. They may not notice that the given factor is the same as one of the factors in #7, and they may or may not have noticed that imaginary roots occur in pairs. This problem is designed to motivate the discussion about this idea. Let students work on the problem, but once they start becoming frustrated, tell them to move on and finish the task. If you have a student that is able to solve #8, he/she should be asked to share during the class discussion.

Discuss:

Begin the discussion with question 3. Ask a student to share that has used long division to divide by the given factor $(x + 3)$ and then factored the quadratic that is left to get the remaining factors. If a student initially graphed the function and then verified the roots, as discussed in the Explore narrative, then ask them to share. Compare the two approaches and discuss how a graph can support the algebraic work. Remind students that if they use a graph to find roots, they must verify them algebraically.

Next, discuss question 5. Ask a previously-selected student to share that has divided by the given factor and then set the quadratic that is left equal to zero and solved by square rooting both sides of the equation. It is a common error to forget to use both the positive and negative roots. If this has happened in class, remind students that this error will cause them to miss one of the roots predicted by the Fundamental Theorem of Algebra.

Discuss #6. The presenting student should have used the quadratic formula to solve the polynomial that remained after dividing by the given factor. This problem provides the opportunity to reinforce how to simplify expressions that result from the quadratic formula.

Briefly discuss question 7. If possible, have a student that has factored the initial polynomial share their work to demonstrate that there is more than one way to solve these types of problems. Be sure that students notice that problem #6 produced a pair of imaginary roots using the quadratic formula, and problem 7 produced a pair of imaginary roots by square rooting both sides of the equation. Tell students these are called **conjugate pairs.**

Move to question 8. If your classroom climate allows students to feel comfortable sharing work that isn't entirely successful, you may want to have a student share their attempt at long division. Ask students if what they have noticed about complex roots in previous problems may be useful in this problem. Use the graph of the function and the Fundamental Theorem of Algebra to argue that there must be another imaginary root. Then show students how to use the conjugate of the given root to find the remaining roots. Verify your work for the class by graphing both the standard form and the factored form to show they are equivalent. Make explicit to students that imaginary roots occur in conjugate pairs.

Finally, wrap up the discussion by completing the table for question 10, showing the possible combinations of real and imaginary roots for each of the function types.

Aligned Ready, Set, Go: *Polynomial Functions 4.4*

READY, SET, GO! Name Period Date

READY

Topic: Ordering numbers from least to greatest

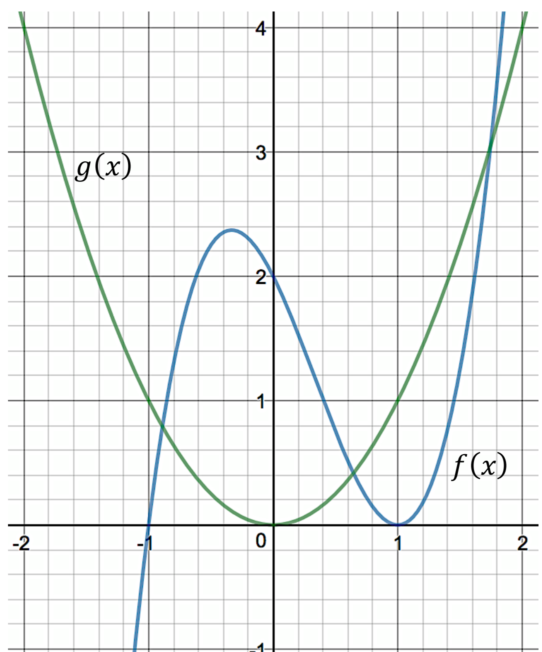
Order the numbers from least to greatest.

1.	100^3	$\sqrt{100}$	$log_2 100$	100	2^{10}
2.	2^{-1}	$-\sqrt{100}$	$log_2\left(\frac{1}{8}\right)$	0	$(-2)^1$
3.	2^0	$\sqrt{25}$	$log_2 8$	$2(x^0), x \neq 0$	$(2)^{-\frac{1}{2}}$
4.	$log_3 3^3$	$log_5 5^{-2}$	$log_6 6^0$	$log_4 4^{-1}$	$log_2 2^3$

Refer to the given graph to answer the questions.

Insert $>, <, or =$ in each statement to make it true.

5. $f(0)$ ________ $g(0)$
6. $f(2)$ ________ $g(2)$
7. $f(-1)$ ________ $g(-1)$
8. $f(1)$ ________ $g(-1)$
9. $f(5)$ ________ $g(5)$
10. $f(-2)$ ________ $g(-2)$



SET

Topic: Finding the roots and factors of a polynomial

Use the given root to find the remaining roots. Then write the function in factored form.

Function	Roots	Factored form
11. $f(x) = x^3 - 13x^2 + 52x - 60$	$x = 5$	
12. $g(x) = x^3 + 6x^2 - 11x - 66$	$x = -6$	
13. $p(x) = x^3 + 17x^2 + 92x + 150$	$x = -3$	
14. $q(x) = x^4 - 6x^3 + 3x^2 + 12x - 10$	$x = \sqrt{2}$	

GO

Topic: Using the distributive property to multiply complex expressions

Multiply using the distributive property. Simplify. Write answers in standard form.

15. $(x-\sqrt{13})(x+\sqrt{13})$

16. $(x-3\sqrt{2})(x+3\sqrt{2})$

17. $(x-4+2i)(x-4-2i)$

18. $(x+5+3i)(x+5-3i)$

19. $(x-1+i)(x-1-i)$

20. $(x+10-\sqrt{2}\,i)(x+10+\sqrt{2}\,i)$

4.5 Is This The End?

A Solidify Understanding Task

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In previous mathematics courses, you have compared and analyzed growth rates of polynomial (mostly linear and quadratic) and exponential functions. In this task, we are going to analyze rates of change and end behavior by comparing various expressions.

Part I: Seeing patterns in **end behavior**

1. In as many ways as possible, compare and contrast linear, quadratic, cubic, and exponential functions.

2. Using the graph provided, write the following functions vertically, from **greatest to least for $\boldsymbol{x = 0}$**. Put the function with the greatest value on top and the function with the smallest value on the bottom. Put functions with the same values at the same level. An example, $l(x) = x^7$, has been placed on the graph to get you started.

$f(x) = 2^x$	$p(x) = x^3 + x^2 - 4$	$g(x) = x^2 - 20$
$h(x) = x^5 - 4x^2 + 1$	$k(x) = x + 30$	$m(x) = x^4 - 1$
$r(x) = x^5$	$n(x) = \left(\frac{1}{2}\right)^x$	$q(x) = x^6$

3. What determines the value of a polynomial function at $x = 0$? Is this true for other types of functions?

4. Write the same expressions on the graph in order from **greatest to least** when x represents a very large number (this number is so large, so we say that it is approaching positive infinity). If the value of the function is positive, put the function in quadrant 1. If the value of the function is negative, put the function in quadrant IV. An example has been placed for you.

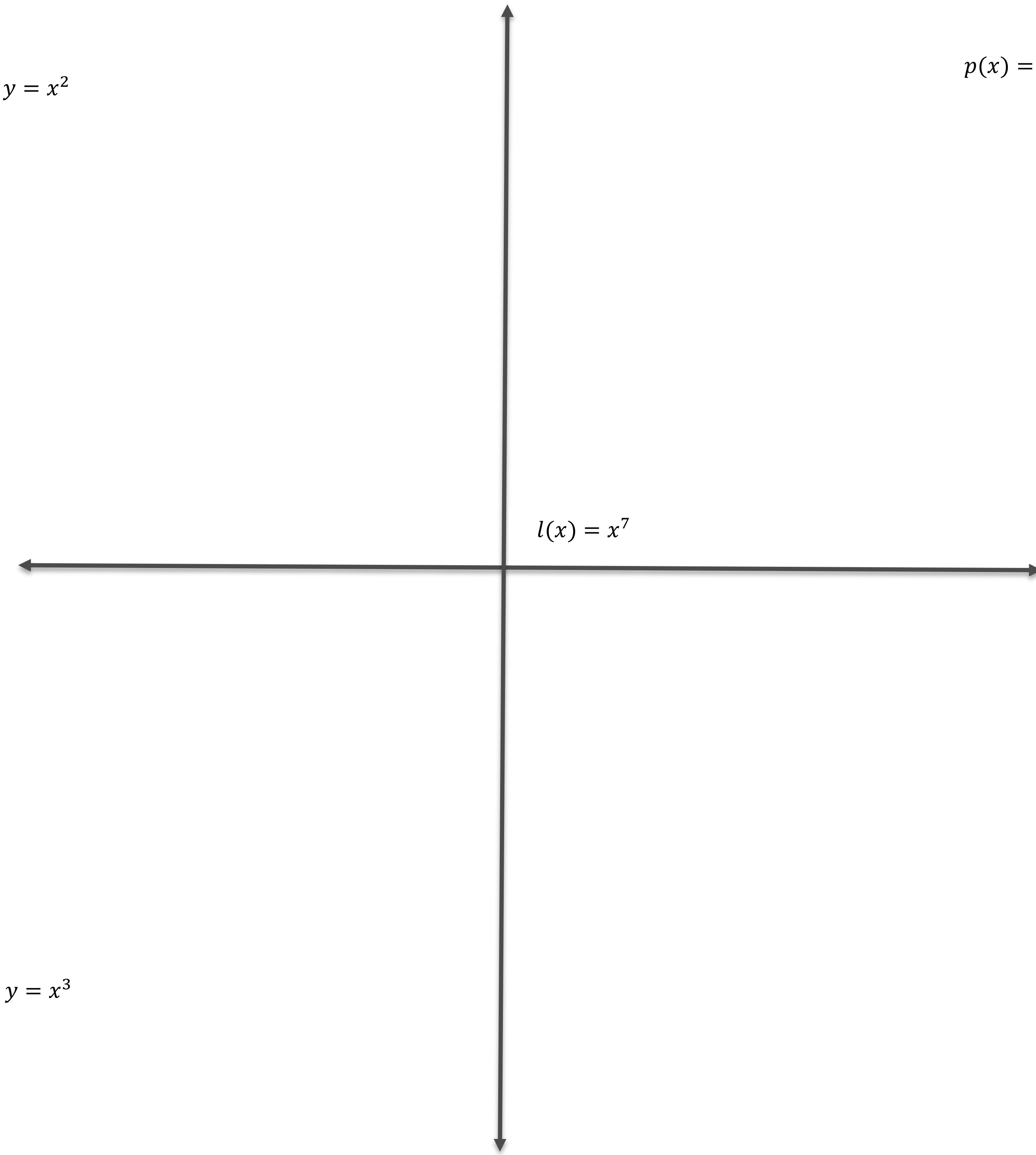
5. What determines the end behavior of a polynomial function for very large values of x?

6. Write the same functions in order from **greatest to least** when x represents a number that is approaching negative infinity. If the value of the function is positive, place it in Quadrant II, if the value of the function is negative, place it in Quadrant III. An example is shown on the graph.

7. What patterns do you see in the polynomial functions for x values approaching negative infinity? What patterns do you see for exponential functions? Use graphing technology to test these patterns with a few more examples of your choice.

8. How would the end behavior of the polynomial functions change if the lead terms were changed from positive to negative?

$\boldsymbol{x \to -\infty}$ $\qquad$ $\boldsymbol{x = 0}$ $\qquad$ $\boldsymbol{x \to \infty}$



x^5	$x^2 - 20$
$x + 30$	$x^4 - 1$
2^x	$\left(\frac{1}{2}\right)^x$
x^7	x^6
$x^3 + x^2 - 4$	
$x^5 - 4x^2 + 1$	

4.5 Is This The End? – Teacher Notes

A Solidify Understanding Task

Purpose: The purpose of this task is to solidify student understanding of how the degree of the polynomial function impacts the rate of change and end behavior. By comparing the values of expressions with 'extreme' values, students will be able to:

- Understand that the degree of the polynomial is the highest valued whole number exponent and that this term determines the end behavior (regardless of the other terms in the expression).
- Determine that the higher the degree of a polynomial, the greater the value as x approaches infinity. Understand that while the highest degree polynomial has the greatest value as $x \to \infty$, that exponential functions have the greatest rate of change, and therefore the greatest value when x becomes very large.
- Identify differences between even and odd degree functions. In this task, students will know the end behavior for odd degree functions (as $x \to -\infty, f(x) \to -\infty$ and that both even and odd degree polynomial functions as $x \to \infty, f(x) \to \infty$) as long as the co-efficient is positive and realize that the opposite is true if the co-efficient is negative.

Teacher Note: Make copies of the expressions and have them cut out in advance. To make the most of the whole group discussion, be prepared by having several 'movable copies' of each expression (large cutouts are attached and being able to move expressions on a Smartboard or using a document camera will be handy). You will want more than one set so that everyone can visually see the change in order depending on the values chosen for x. See Discuss section for more details.

Core Standards Focus:

F.LE.3 Observe using graphs and tables that a quantity increasing exponentially eventually exceeds a quantity increasing linearly, quadratically, or (more generally) as a polynomial function.

A.SSE.1 Interpret expressions that represent a quantity in terms of its context.

a. Interpret parts of an expression, such as terms, factors, and coefficients.

F.IF.4 For a function that models a relationship between two quantities, interpret key features of graphs and tables in terms of the quantities, and sketch graphs showing key features given a verbal description of the relationship. *Key features include: intercepts; intervals where the function is increasing, decreasing, positive, or negative; relative maximums and minimums; symmetries; end behavior; and periodicity.*

F.BF.3 Identify the effect on the graph of replacing *f(x) by f(x)+k, kf(x), f(kx), and f(x+k)* for specific values of k (both positive and negative); find the value of *k* given the graphs. Experiment with cases and illustrate an explanation of the effects on the graph using technology. Include recognizing even and odd functions from the graphs and algebraic expressions for them.

Related Standards: F.IF.7

Standards for Mathematical Practice:

- **SMP 1 – Make sense of problems and persevere in solving them**
- **SMP 3 – Construct viable arguments and critique the reasoning of others**
- **SMP 5 – Use appropriate tools strategically**
- **SMP 7 – Look for and make use of structure**

Vocabulary: **end behavior**

The Teaching Cycle:

Launch (Whole Class):

Begin by reading the introduction in the task and then explain to students that while they will be comparing and ordering expressions, they should also pay attention to *why* the expressions are in a particular order. Tell students that they should be looking for patterns that they can generalize and use.

Explore (Small Group):

Start the task by having students work individually and then work together in small groups on the rest of the task. As you monitor, listen for student reasoning throughout, paying particular attention when they are discussing the order of the expressions in questions 3 and 4 (connecting to end behavior). Take note of how different groups are determining order. Are they plugging in values? Are they looking at a graphical representation? Are they making assumptions based on their knowledge of exponents? As students answer question 5, press for them to explain their reasoning. Listen for answers that include comparing polynomials to exponential functions (and that exponential functions with a base greater than one will always outgrow any polynomial function). Also listen for answers that distinguish between when the degree of the polynomial is even versus when it is odd (and how this impacts the values as $x \rightarrow -\infty$).

Discuss (Whole Class):

When all students have had the opportunity to answer question 6 (some may have already completed all questions), move to the whole group discussion. Begin with a student that shares their ordering for question 2. Ask the student to explain how they figured out the order. During this explanation, there should be some discussion of substituting in 0 for x and finding that the only term left in a polynomial is the constant term. Ask the class if this will be the case for every polynomial function. Ask the class if this is true of all functions. They should recognize that some functions, like logarithms are undefined at $x = 0$, and some functions, like exponentials, depend on more than just a constant term.

Next, ask a student to share their ordering of the functions for #4. Use either the axes provided in the task or post a larger set of axes and use the large function cards, that are provided following the teacher notes, so that the order is very visual for students. Focus first on the order of the polynomial functions. Be sure that students notice that the highest-powered term essentially determines the end behavior for large values of x. Look specifically at h(x) and r(x). Since these two functions are both degree 5 polynomials, then they are more difficult to order. Ask for students to share different ideas for how to compare these two functions, including using technology to graph both functions and comparing them for large values, substituting large numbers into both functions and comparing, and reasoning about the effect of the additional terms in h(x). It is likely that there will be some controversy around whether or not the exponential function is the greatest as $x \to \infty$. Look for similar strategies to compare the exponential with the highest-powered polynomial. Then, ask students to consider the rates of change of the two functions to think about which one is the greatest for large values of x. Have students calculate the average rate of change of both functions from [15, 20] to see that the rapid rate of change of the exponential function will eventually make it exceed any polynomial.

Next, ask a student to share their ordering of the functions for question #6. Again, make the visual available so that all students can easily see the pattern that even-powered polynomials approach infinity as $x \to -\infty$ and odd-powered polynomials approach negative infinity as $x \to -\infty$. Ask students if this is consistent with their experience of quadratic and cubic functions, and why it would be so. Ask how this connects to the number and types of roots that were found for polynomial functions in the previous task. (Be prepared to pull the chart out as part of the discussion.) Also discuss question #8 and the effect of changing the sign of the lead coefficient in a polynomial.

Ask students to share as many of the problems in Part II as time allows. Be sure that they are using the results of Part I and the correct notation to record their results.

Finally ask students for their observations about even functions. Highlight observations that include the idea that the graphs are symmetric about the y-axis. Ask students how the definition given relates to the symmetry of the graphs and help them to articulate that the output of the function is the same for both positive and negative values of x in even functions. This is true of the parent function for all even-degree polynomials and if they have a vertical translation and/or a vertical stretch.

Repeat the process by asking students for their observations about odd functions. It may be harder to recognize the 180˚ rotation about the origin. Demonstrating this idea with a piece of patty paper may help students to see it more easily. Reinforce how the definition of an odd function creates this pattern in the graph of odd functions. Ask students how to tell if an odd-degree polynomial will be an odd function. Support them to think about which transformations of an odd-degree parent function like $f(x) = x^3$ will result in an odd function. Close the discussion by asking students to share a couple of examples of functions that are neither odd nor even. Many of the examples from earlier in the task can be used for this purpose.

Aligned Ready, Set, Go: *Polynomial Functions 4.5*

READY, SET, GO! Name Period Date

READY

Topic: Recognizing special products

Multiply.

1. $(x + 5)(x + 5)$
2. $(x - 3)(x - 3)$
3. $(a + b)(a + b)$

4. In problems 1 – 3 the answers are called **perfect square trinomials**. What about these answers makes them be a **perfect square trinomial**?

5. $(x + 8)(x - 8)$
6. $(x + \sqrt{3})(x - \sqrt{3})$
7. $(x + b)(x - b)$

8. The products in problems 5 – 7 end up being binomials, and they are called the **difference of two squares**. What about these answers makes them be the **difference of two squares**?

   Why don't they have a middle term like the problems in 1 – 3?

9. $(x - 3)(x^2 + 3x + 9)$
10. $(x + 10)(x^2 - 10x + 100)$
11. $(a + b)(a^2 - ab + b^2)$

12. The work in problems 9 – 11 makes them feel like the answers are going to have a lot of terms. What happens in the work of the problem that makes the answers be binomials?

    These answers are called the **difference of two cubes** (#9) and the **sum of two cubes** (#10 and #11.) What about these answers makes them be the **sum or difference of two cubes**?

SET

Topic: Determining values of polynomials at zero and at $\pm\infty$. (End behavior)

State the y-intercept, the degree, and the end behavior for each of the given polynomials.

13. $f(x) = x^5 + 7x^4 - 9x^3 + x^2 - 13x + 8$

y- intercept:

Degree:

End behavior:

As $x \to -\infty,\ f(x) \to$ ________

As $x \to +\infty,\ f(x) \to$ ________

14. $g(x) = 3x^4 + x^3 + 5x^2 - x - 15$

y- intercept:

Degree:

End behavior:

As $x \to -\infty,\ g(x) \to$ ________

As $x \to +\infty,\ g(x) \to$ ________

15. $h(x) = -7x^9 + x^2$

y- intercept:

Degree:

End behavior:

As $x \to -\infty,\ h(x) \to$ ________

As $x \to +\infty,\ h(x) \to$ ________

16. $p(x) = 5x^2 - 18x + 4$

y- intercept:

Degree:

End behavior:

As $x \to -\infty,\ p(x) \to$ ________

As $x \to +\infty,\ p(x) \to$ ________

17. $q(x) = x^3 - 94x^2 - x - 20$

y- intercept:

Degree:

End behavior:

As $x \to -\infty,\ q(x) \to$ ________

As $x \to +\infty,\ q(x) \to$ ________

18. $y = -4x + 12$

y- intercept:

Degree:

End behavior:

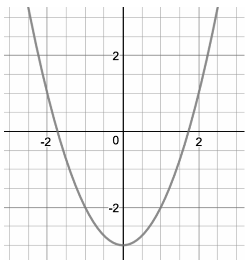
As $x \to -\infty,\ y \to$ ________

As $x \to +\infty,\ y \to$ ________

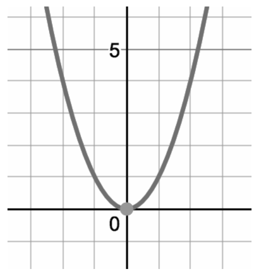
Topic: Identifying even and odd functions

19. Identify each function as even, odd, or neither.

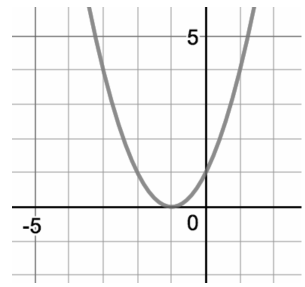
a) $f(x) = x^2 - 3$



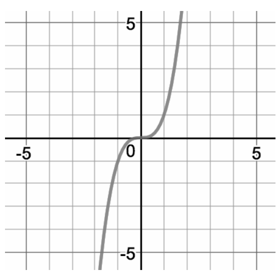
b) $f(x) = x^2$



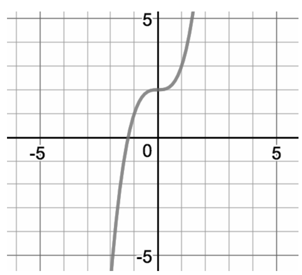
c) $f(x) = (x+1)^2$



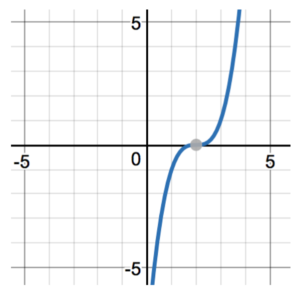
d) $f(x) = x^3$



e) $f(x) = x^3 + 2$



f) $f(x) = (x-2)^3$



GO

Topic: Factoring special products

Fill in the blanks on the sentences below.

20. The expression $\boldsymbol{a^2 + 2ab + b^2}$ is called a **perfect square trinomial**. I can recognize it because the first and last terms will always be perfect ______________________________.

The middle term will be 2 times the _____________________ and ______________________.

There will always be a _____________ sign before the last term.

It factors as (_____________)(_____________).

21. The expression $\boldsymbol{a^2 - b^2}$ is called the **difference of 2 squares**. I can recognize it because it's a binomial and the first and last terms are perfect ______________________________.
The sign between the first term and the last term is always a __________________ .
It factors as (___________)(___________).

22. The expression $\boldsymbol{a^3 + b^3}$ is called the **sum of 2 cubes**. I can recognize it because it's a binomial and the first and last terms are ______________________. The expression $a^3 + b^3$ factors into a binomial and a trinomial. I can remember it as a *short* (___) and a *long* (_________).
The sign between the terms in the binomial is the ____________ as the sign in the expression. The first sign in the trinomial is the _______________ of the sign in the binomial. That's why all of the middle terms cancel when multiplying.
The last sign in the trinomial is always _______.
It factors as (___________)(_____________________).

Factor using what you know about special products.

23. $25x^2 + 30 + 9$

24. $x^2 - 16$

25. $x^3 + 27$

26. $49x^2 - 36$

27. $x^3 - 1$

28. $64x^2 - 240 + 225$

Part II: Using end behavior patterns

For each situation:

- Determine the function type. If it is a polynomial, state the degree of the polynomial and whether it is an even degree polynomial or an odd degree polynomial.
- Describe the end behavior based on your knowledge of the function. Use the format: As $x \to -\infty, f(x) \to$ _____ $and \quad as\ x \to \infty\ f(x) \to$ _____

1. $f(x) = 3 + 2x$

   Function type:

   End behavior: As $x \to -\infty, f(x) \to$ _____

   End behavior: As $x \to \infty, f(x) \to$ _____

2. $f(x) = x^4 - 16$

   Function type:

   End behavior: As $x \to -\infty, f(x) \to$ _____

   End behavior: As $x \to \infty, f(x) \to$ _____

3. $f(x) = 3^x$

   Function type:

   End behavior: As $x \to -\infty, f(x) \to$ _____

   End behavior: As $x \to \infty, f(x) \to$ _____

4. $f(x) = x^3 + 2x^2 - x + 5$

   Function type:

   End behavior: As $x \to -\infty, f(x) \to$ _____

   End behavior: As $x \to \infty, f(x) \to$ _____

5. $f(x) = -2x^3 + 2x^2 - x + 5$

   Function type:

   End behavior: As $x \to -\infty, f(x) \to$ _____

   End behavior: As $x \to \infty, f(x) \to$ _____

6. $f(x) = log_2 x$

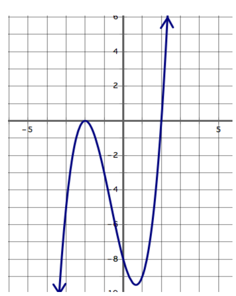
   Function type:

   End behavior: As $x \to -\infty, f(x) \to$ _____

   End behavior: As $x \to \infty, f(x) \to$ _____

Use the graphs below to describe the end behavior of each function by completing the statements.

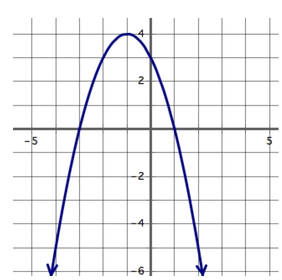
7.



End behavior: As $x \to -\infty, f(x) \to$ _____

End behavior: As $x \to \infty, f(x) \to$ _____

8.



End behavior: As $x \to -\infty, f(x) \to$ _____

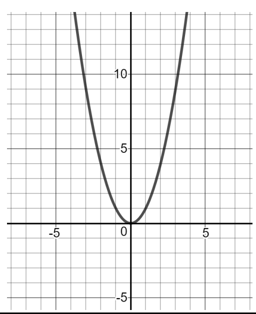
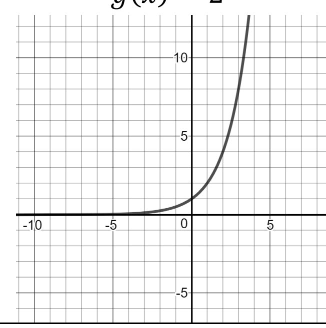
End behavior: As $x \to \infty, f(x) \to$ _____

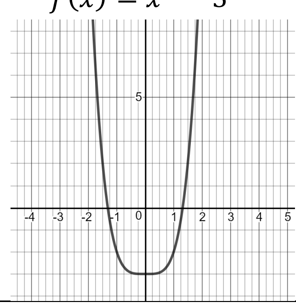
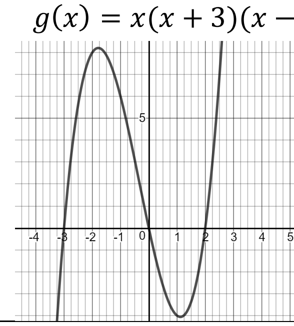
9. How does the end behavior for quadratic functions connect with the number and type of roots for these functions? How does the end behavior for cubic functions connect with the number and type of roots for cubic functions?

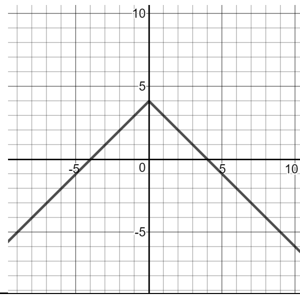
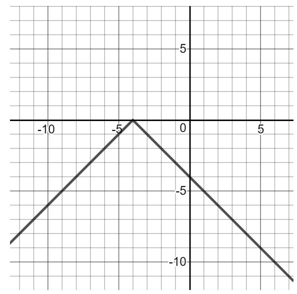
Part III: Even and Odd Functions

Some functions that are not polynomials may be categorized as even functions or odd functions. When mathematicians say that a function is an even function, they mean something very specific.

1. Let's see if you can figure out what the definition of an even function is with these examples:

Even function: $f(x) = x^2$ 	Not an even function: $g(x) = 2^x$ 
Differences:	

Even function: $f(x) = x^4 - 3$ 	Not an even function: $g(x) = x(x+3)(x-2)$ 
Differences:	

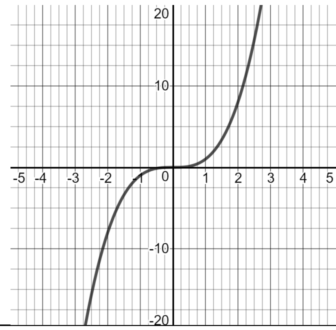
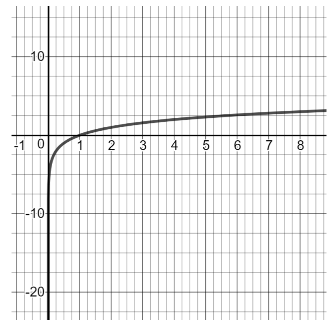
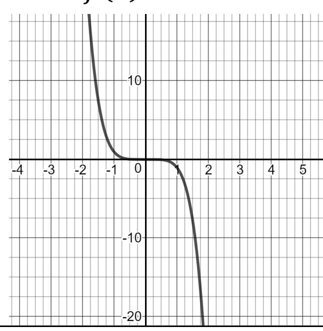
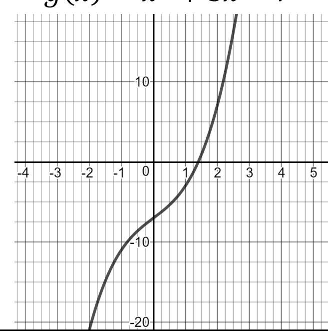
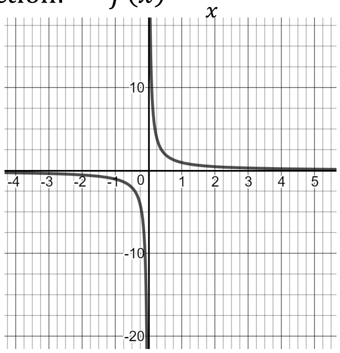
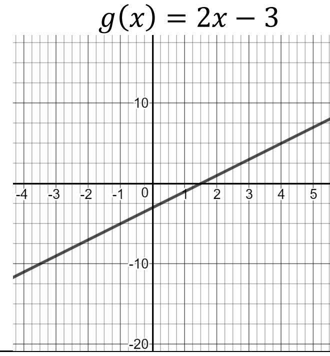
<table>
<tr><td>Even function:
$f(x) = -|x| + 4$

</td><td>Not an even function:
$g(x) = -|x + 4|$

</td></tr>
<tr><td colspan="2">Differences:</td></tr>
<tr><td>Even function:
$f(2) = 5 \ and \ f(-2) = 5$</td><td>Not an even function:
$g(2) = 3 \ and \ g(-2) = 5$</td></tr>
<tr><td colspan="2">Differences:</td></tr>
</table>

2. What do you observe about the characteristics of an even function?

3. The algebraic definition of an even function is:
   $f(x)$ is an even function if and only if $f(x) = f(-x)$ for all values of x in the domain of f.
   What are the implications of the definition for the graph of an even function?

4. Are all even-degree polynomials even functions? Use examples to explain your answer.

5. Let's try the same approach to figure out a definition for odd functions.

<table>
<tr><td>Odd function:
$f(x) = x^3$

</td><td>Not an odd function:
$g(x) = \log_2 x$

</td></tr>
<tr><td colspan="2">Differences:</td></tr>
<tr><td>Odd function:
$f(x) = -x^5$

</td><td>Not an odd function:
$g(x) = x^3 + 3x - 7$

</td></tr>
<tr><td colspan="2">Differences:</td></tr>
<tr><td>Odd function: $f(x) = \frac{1}{x}$

</td><td>Not an odd function:
$g(x) = 2x - 3$

</td></tr>
<tr><td colspan="2">Differences:</td></tr>
<tr><td>Odd function:
$f(2) = 3 \; and \; f(-2) = -3$</td><td>Not an odd function:
$g(2) = 3 \; and \; g(-2) = 5$</td></tr>
<tr><td colspan="2">Differences:</td></tr>
</table>

6. What do you observe about the characteristics of an odd function?

7. The algebraic definition of an odd function is:

$f(x)$ is an odd function if and only if $f(-x) = -f(x)$ for all values of x in the domain of f.

Explain how each of the examples of odd functions above meet this definition.

8. How can you tell if an odd-degree polynomial is an odd function?

9. Are all functions either odd or even?

4.6 Puzzling Over Polynomials

A Practice Understanding Task

For each of the polynomial puzzles below, a few pieces of information have been given. Your job is to use those pieces of information to complete the puzzle. Occasionally, you may find a missing piece that you can fill in yourself. For instance, although some of the roots are given, you may decide that there are others that you can fill in.

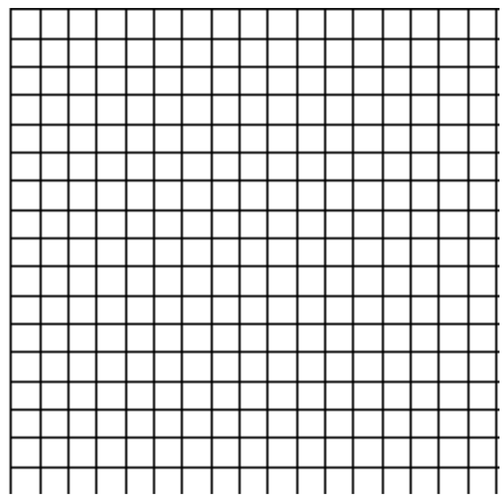
<table>
<tr>
<td>1.</td>
<td>Function (in factored form)

Function (in standard form)

End behavior:
$as \quad x \to -\infty, \qquad f(x) \to$ ____
$as \quad x \to \infty, \qquad f(x) \to$ ____

Roots (with multiplicity):
-2, 1, and 1

Value of leading co-efficient:
-2

Degree: 3</td>
<td>Graph:
</td>
</tr>
</table>

<table>
<tr>
<td>2.</td>
<td>Function (in factored form)

Function (in standard form)

End behavior:
$as \quad x \to -\infty, \qquad f(x) \ \to$ ____
$as \quad x \to \ \ \infty, \qquad f(x) \ \to$ ____

Roots (with multiplicity):
$2 + i, 4, 0$

Value of leading co-efficient:
1

Degree: 4</td>
<td>Graph:</td>
</tr>
</table>

<table>
<tr>
<td>3.</td>
<td>Function:
$f(x) = 2(x-1)(x+3)^2$

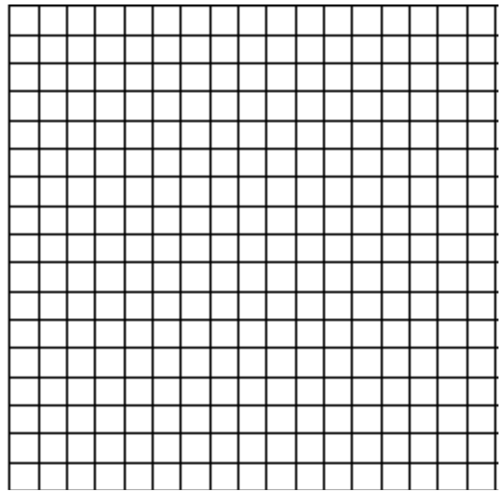
End behavior:
$as \quad x \to -\infty, \qquad f(x) \ \to$ ____
$as \quad x \to \ \ \infty, \qquad f(x) \ \to$ ____

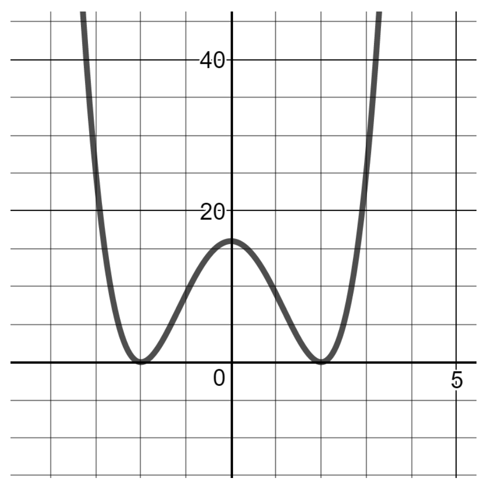
Roots (with multiplicity):

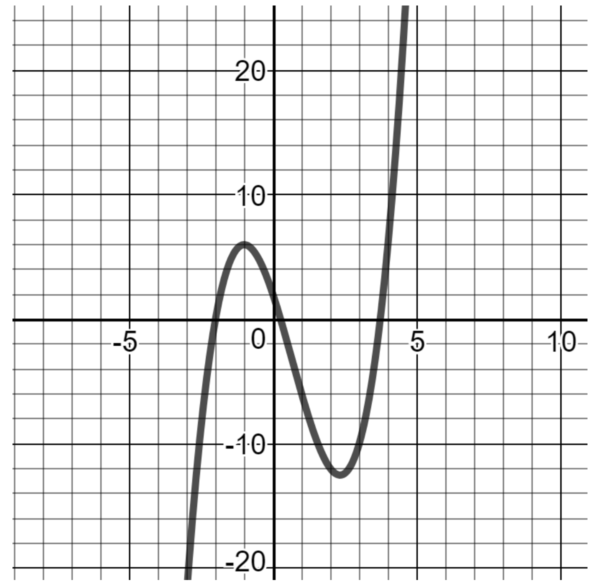
Value of leading co-efficient:

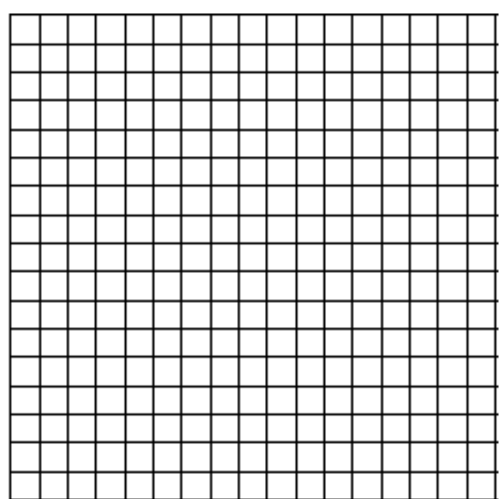
Domain:

Range: All Real numbers</td>
<td>Graph:</td>
</tr>
</table>

4.	**Function:** **End behavior:** $as \quad x \to -\infty, \qquad f(x) \ \to \ \infty$ $as \quad x \to \ \ \infty, \qquad f(x) \ \to$ ____ **Roots (with multiplicity):** (3,0) m: 1; (-1,0) m: 2 (0,0) m: 2 **Value of leading co-efficient:** -1 **Domain:** **Range:**	Graph: 

5.	**Function:** **End behavior:** $as \quad x \to -\infty, \qquad f(x) \ \to$ ____ $as \quad x \to \ \ \infty, \qquad f(x) \ \to$ ____ **Roots (with multiplicity):** **Value of leading co-efficient:** 1 **Domain:** **Range:** **Other:** $f(0) = 16$	Graph: 

6.	**Function (in standard form):** $f(x) = x^3 - 2x^2 - 7x + 2$ **Function (in factored form):** **End behavior:** $as\ \ x \to -\infty, \quad f(x) \to$ ____ $as\ \ x \to \infty, \quad f(x) \to$ ____ **Roots (with multiplicity):** -2 **Domain:** **Range:**	Graph: 

7.	**Function (in standard form):** $f(x) = x^3 - 2x$ **Function (in factored form):** **End behavior:** $as\ \ x \to -\infty, \quad f(x) \to$ ____ $as\ \ x \to \infty, \quad f(x) \to$ ____ **Roots (with multiplicity):** **Domain:** **Range:**	Graph: 

4.6 Puzzling Over Polynomials – Teacher Notes

A Practice Understanding Task

Purpose: The purpose of this task is tie together what students have learned about roots, end behavior, operations, and graphs of polynomials. Each problem gives a few features of a polynomial and asks students to find other features, sometimes including the graphs. This will require them to know the end behavior of a polynomial, based on the degree, and to find all the roots, given some of the roots. In most cases, students are asked to write the equation of the polynomial or to write the equation in a different form than is given.

Core Standards Focus:

F-IF.7: Graph functions expressed symbolically and show key features of the graph, by hand in simple cases and using technology for more complicated cases.★

a. Graph linear and quadratic functions and show intercepts, maxima, and minima.

c. Graph polynomial functions, identifying zeros when suitable factorizations are available, and showing end behavior. (3.2,

A-APR.3: Identify zeros polynomials when suitable factorizations are available, and use the zeros to construct a rough graph of the function defined by the polynomial.

N-CN.8: Extend polynomial identities to the complex numbers. *For example, rewrite* $x^2 + 4\ as\ (x + 2i)(x - 2i)$.

N-CN.9: Know the Fundamental Theorem of Algebra; show that it is true for quadratic polynomials. (In Secondary III, limit to polynomials with real coefficients).

A-CED: Create equations that describe numbers or relationships

2. Create equations in two or more variables to represent relationships between quantities; graph equations on coordinate axes with labels and scales.

Standards for Mathematical Practice:

SMP 1 – Make sense of problems and persevere in solving them

SMP 6 – Attend to precision

The Teaching Cycle:

Launch:

Launch the task by telling students that they are given several polynomial puzzles that will help them to connect all the work that they have done in the module so far. In each puzzle they are given some information and they need to use that information to find the other information. Sometimes the puzzles are little sneaky because information that they need is not given directly, but if they think about what they know, they can find it.

Explore:

Monitor students as they work to see which problems are causing difficulty so that they can be used in the class discussion. You may need to support students in solving problem 2. One imaginary root is given, and they will need to know that the other root is the conjugate. If students are stuck, ask questions to help remind them of the relationship between imaginary roots of a polynomial. As students work on problems like #5 and #6, allow them to use the given graph to find the integer roots and then require them to use those roots to find the remaining irrational or imaginary roots using long division.

Discuss:

Discuss as many of the problems as time allows, with previously-selected students sharing their work for each problem. Start with the problems that were more difficult so that students have a chance to see how to work through some of the difficulties. After a student presents their work, ask the rest of the class to summarize the process that the presenting student used with the following sentence frame:

In the problem, we were given ______________________________ and so we knew ___________________.

Then we were able to find _____________________________ by ____.

An example of using the sentence frame with problem #2 is:

In the problem, we were given **2 real roots and 1 imaginary root** and so we knew **there was another imaginary root, $2 - i$.**

Then we were able to find **the equation** by **writing each root as a factor and multiplying them together.**

Aligned Ready, Set, Go: ***Polynomial Functions 4.6***

READY, SET, GO! Name Period Date

READY

Topic: Reducing rational numbers and expressions

Reduce the expressions to lowest terms. (Assume no denominator equals 0.)

1. $\frac{3x}{6x^2}$

2. $\frac{2 \cdot 5 \cdot x \cdot x \cdot x \cdot y}{3 \cdot 5 \cdot x \cdot y \cdot y}$

3. $\frac{7ab^2}{7ab^2}$

4. $\frac{(x+2)(x-9)}{(x+2)(x-9)}$

5. $\frac{(3x-5)(x+4)}{(x-1)(3x-5)}$

6. $\frac{(2x-11)(3x+17)}{(2x-11)(3x-5)}$

7. $\frac{(8x-7)(x+3)}{8x(x+3)(2x-3)}$

8. $\frac{3x(2x+7)(x-1)(6x-5)}{x(2x+7)(x-1)(6x-5)}$

9. Why is it important that the instructions say to assume that no denominator equals 0?

SET

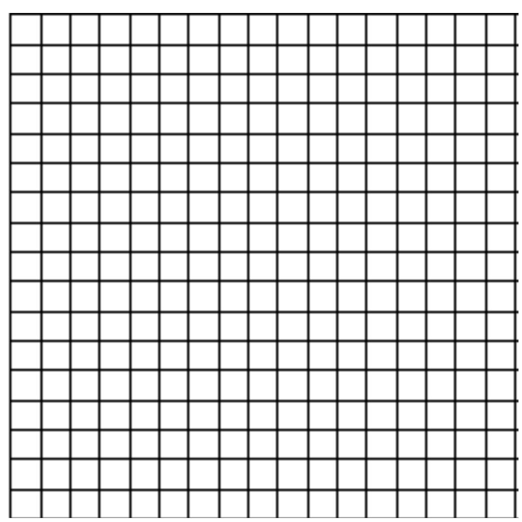
Topic: Reviewing features of polynomials

Some information has been given for each polynomial. Fill in the missing information.

10.

Graph:

Function: $f(x) = x^3$



Function in factored form:

End behavior:

As $x \to -\infty, f(x) \to$ ______ As $x \to \infty, f(x) \to$ _____

Roots (with multiplicity):

Degree:

Value of leading co-efficient:

11.
Graph:
Function in standard form:

Function in factored form: $g(x) = -x(x-2)(x-4)$

End behavior:
As $x \to -\infty, g(x) \to$ ______ As $x \to \infty, g(x) \to$ _____

Roots (with multiplicity):

Degree:

Value of leading co-efficient:

12.
Graph:
Function in standard form: $h(x) = x^3 - 2x^2 - 3x$

Function in factored form:

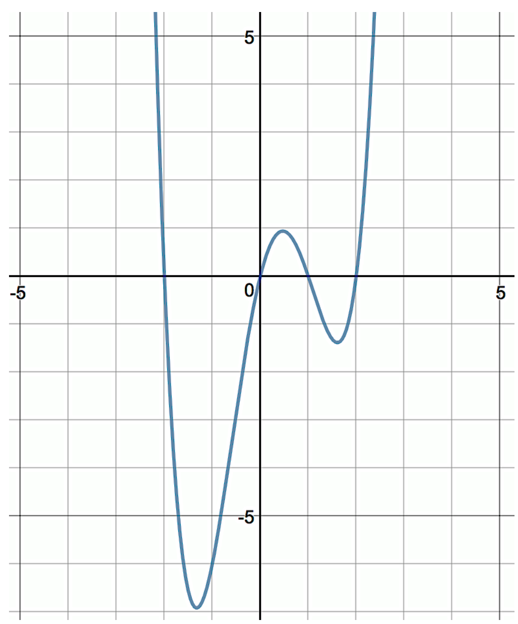
End behavior:
As $x \to -\infty, h(x) \to$ ______ As $x \to \infty, h(x) \to$ _____

Roots (with multiplicity):

Degree:

Value of $h(2)$:

13.
Graph:



Function in standard form:

Function in factored form:

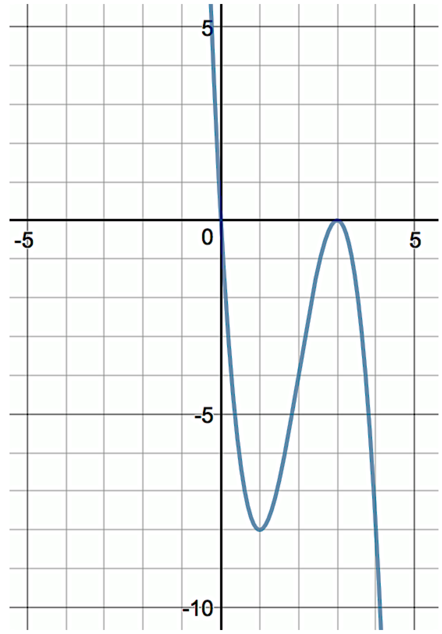
End behavior:
As $x \to -\infty, f(x) \to$ ______ As $x \to \infty, f(x) \to$ _____

Roots (with multiplicity):

Degree:

y-intercept:

14. **Graph:**



Function in standard form:

Function in factored form:

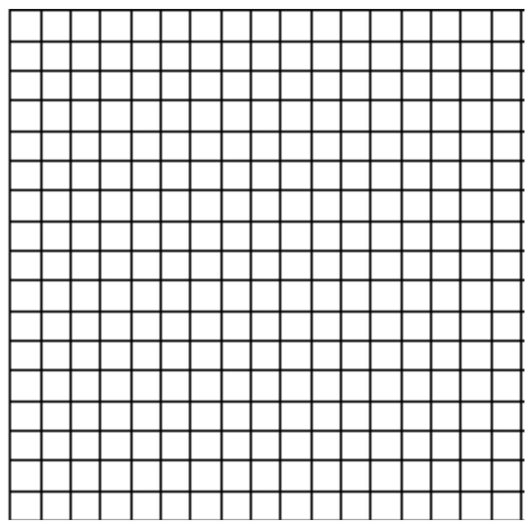
End behavior:
As $x \to -\infty, p(x) \to$ ______ As $x \to \infty, p(x) \to$ _____

Roots (with multiplicity):

Degree:

Value of leading coefficient:

15. **Graph:**



Function in standard form: $q(x) = x^3 + 2x^2 + x + 2$

Function in factored form:

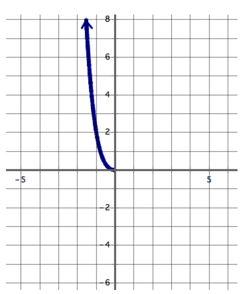
End behavior:
As $x \to -\infty, q(x) \to$ ______ As $x \to \infty, q(x) \to$ _____

Roots (with multiplicity):
$x = i$

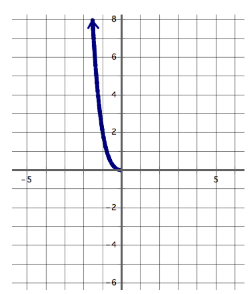
Degree:

y-intercept:

16. Finish the graph if it is an even function.



17. Finish the graph if it is an odd function.



GO

Topic: Writing polynomials given the zeros and the leading coefficient

Write the polynomial function in standard form given the leading coefficient and the zeros of the function.

18. Leading coefficient: 2; $roots$: $2, \sqrt{2}, -\sqrt{2}$

19. Leading coefficient: -1; $roots$: $1, 1+\sqrt{3},\ 1-\sqrt{3}$

20. Leading coefficient: 2; $roots$: $4i, -4i$

Fill in the blanks to make a true statement.

21. If $f(b) = 0$, then a factor of $f(b)$ must be ______________________________.

22. The rate of change in a linear function is always a ________________________________.

23. The rate of change of a quadratic function is ____.

24. The rate of change of a cubic function is ____.

25. The rate of change of a polynomial function of degree n can be described by a function of degree ____________________.