Using the *Comprehensive Mathematics Instruction* (CMI) Framework to Analyze a Mathematics Teaching Episode

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<u>Introduction</u>

Teaching and learning are processes that are so interdependent, it is difficult to study one without considering the other. Ideally, not only do teachers plan their instruction with their specific students in mind, but they also assess their students' learning in the act of instruction and modify their actions as needed to help their students achieve the desired learning outcomes. One purpose of the *Comprehensive Math Instruction* (CMI) Framework is to highlight this interconnection between teaching and learning in order to help teachers teach for deep mathematical understanding. Specifically, the CMI Framework can be used by the classroom teacher as a pedagogical tool before, during, and after teaching.

Prior to teaching, the Framework provides a planning model for designing lessons to meet intended purposes and desired learning outcomes. In the act of teaching, the Framework provides access for teachers to see order within the chaos of overwhelming and seemingly spontaneous classroom events, as well as a means to analyze the ideas that are present in the classroom and to plan productive and appropriate responses. When reviewing a teaching episode, the Framework creates opportunities for reflection by providing a lens through which to view the instruction. The Framework also focuses teachers on future work—where they can go next with student thinking and suggesting possible paths for how to get there.

While the CMI Framework consists of three major components: a *Teaching Cycle*, a *Learning Cycle*, and a *Continuum of Mathematical Understanding*, in this paper we will focus only on the Teaching and Learning Cycles.

<u>The Teaching Cycle</u>

Successful inquiry-based teaching moves through stages of a *Teaching Cycle* (Figure 1) that begins by engaging students in a worthwhile mathematical task (*Launch*), allows students time to grapple with the mathematics of the task (*Explore*), and concludes with a class discussion in which student thinking is examined and exploited for its potential learning opportunities (*Discuss*).

Figure 1: The Teaching Cycle

<u>The Learning Cycle</u>

We conceptualize that student learning progresses through phases of a *Learning Cycle* (Figure 2) that first surfaces students' thinking relative to a selected mathematical purpose (*Develop Understanding*), then extends and solidifies correct and relevant thinking (*Solidify Understanding*), and finally refines thinking in order to acquire fluency consistent with the mathematical community of practice both inside and outside the classroom (*Practice Understanding*).

Figure 2: The Learning

Interactions Between the Teaching and Learning Cycles: An Example

In the CMI Framework each phase of the *Learning Cycle* is fleshed out with detailed descriptions of the *Launch, Explore* and *Discuss* stages of the *Teaching Cycle*. These descriptions include three components: a purpose statement, the teacher role, and the student role. While the three stages of the *Teaching Cycle* are modified by each phase of the *Learning Cycle*, in this paper we will focus on the *Develop Understanding* phase of the *Learning Cycle* to illustrate how the CMI Framework can be useful to a teacher in planning and guiding instruction.

When planning a *Develop Understanding* lesson, the teacher should identify a mathematical purpose for the lesson aligned with state or national standards and select or design an appropriate task to surface student thinking relative to the chosen mathematical purpose. The teacher should also anticipate possible student thinking generated by the task for two reasons: 1) to prepare possible questions that will promote student exploration and discourse during the *Explore* stage of the lesson, and 2) to plan a possible structure and flow of the whole group discussion during the *Discuss* stage. Finally, when planning a lesson, the teacher should determine which specific student groupings (individuals, pairs, or small groups) will best promote learning of the mathematical purpose during the exploration.

During the *Launch* stage of instruction, the teacher activates students' background knowledge and clarifies the task. During the *Explore* stage, the teacher facilitates student exploration and discourse by asking questions to engage students in the task, to prompt student exploration, and to clarify and deepen mathematical thinking. Also, during the *Explore* stage, the teacher is formatively assessing the student work in order to select ideas, strategies and/or representations to share during the *Discuss* stage of the lesson. This selection may include incorrect examples of student work in order to illustrate common misconceptions among students within the class.

The purpose of the Discuss stage is to develop student understanding of emerging ideas, strategies, and representations by having students communicate, explain and support their own thinking and interact with the thinking of their peers. During this stage the teacher orchestrates a discussion by purposefully selecting relevant examples of student work related to the mathematical purpose of the lesson. This type of discourse is more challenging than simply inviting all students to share their work. Instead the teacher helps students understand criteria for judging the emerging mathematics and helps them clarify the mathematical reasoning behind it. The teacher also helps students compare and connect the various ideas, strategies and/or representations under discussion, providing appropriate mathematical vocabulary as needed.

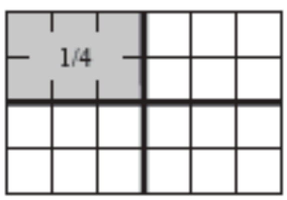
Analysis of a *Develop Understanding* Lesson

For the remainder of this paper, we analyze a teaching episode using the Framework as a tool for making sense of the work of the students and the teacher in the classroom. Monte, a fourth grade teacher, is beginning a series of lessons focused on using fractions to name a portion of a set of objects and has designed this lesson using the Framework as a planning guide. Prior to this lesson, his students have solidified the area concept of fraction and have practiced creating representations of fractions by shading a portion of a region (such as, identifying ¼ of a rectangle by dividing the rectangle into four congruent parts and shading one of them). This *solidified concept* and *practiced representation* become the "inputs" into a new cycle of lessons designed to surface, solidify and practice a new way of thinking about fractions—that a fraction can be used to name a portion of a set of objects.

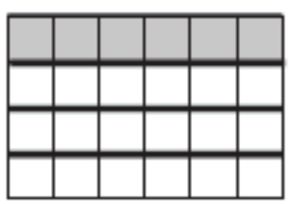
In this lesson, Monte expects students will see new ways of shading a rectangular array to represent a fractional amount and begin to notice that fractions can also be used to name a portion of a set. Intending that this lesson will surface new ideas about fractions and new ways of representing fractional parts of a region, Monte has characterized his class as being in the *Developing Understanding* phase of learning. Because Monte is aware that his goal during this phase of the learning cycle is to develop understanding, we note that he does not attempt to solidify all the ideas that arise during the discussion, knowing that an attempt to do so might "shut down" the broad perspectives he hopes students will bring to the work. Instead, we observe that during the implementation of the lesson Monte makes note of ideas that can be re-examined and solidified in future lessons.

Monte launched his lesson by talking about how hungry he was the previous day and that he had planned to eat ¼ of a cake when he got home. However, when he arrived home he found that his wife had already cut the cake into smaller pieces, forming a 4 ′ 6 rectangular array, as shown in an image which Monte projected onto the whiteboard. Since the cake is already cut into smaller pieces, Monte wonders how he can decide how much to eat so he will still eat ¼ of the cake.

At this point Monte distributed a sheet with several 4 ′ 6 rectangular arrays on it to each student and gave them the task of illustrating different ways he could eat ¼ of the cake. While this representation of fraction as area has the potential to develop students' ideas about fraction as objects within a set, within a matter of just a few seconds all students drew the following representation on the paper, and most seemed content that they had finished the task.



A few students made a second drawing that looked like this.



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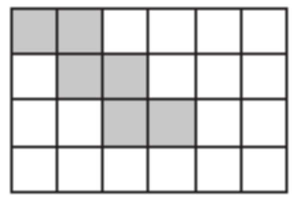
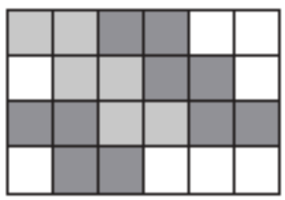
Monte realized that the lack of variety in student responses could have been a result of at least two different reasons. First, the specific phrasing of the question used to launch the task may not have prompted the students to explore multiple solutions, or second, the students' conceptualization of fractions may be so closely tied to area, that they couldn't see noncontiguous pieces combining to form a fraction of the set. Determining the reason behind the lack of variety in student solutions will inform Monte's instructional decisions regarding how to move his students towards the mathematical purpose of the lesson.

Even though the *Explore* stage only lasted a few minutes, the students were definitely finished. In order to determine the reason for the limited responses, Monte decided to move to the *Discuss* phase to have students present their drawings and explain how they knew that their shaded regions represented ¼ of the cake. Monte is aware that one of the students, Lars, has a unique representation that could move the class towards the mathematical purpose; however, Monte first selects the most common representations to give students an opportunity to explain their thinking.

Andrew described his work on the first diagram. "Use imaginary lines to make 4 big parts. Then eat one of the big parts." Andrew's language suggests that he recognizes that ¼ of the cake was made up of smaller pieces. "Or," said Megan, "you can divide the cake up into parts like this," as she pointed to the second diagram. Despite Andrew's use of the phrase "big parts", it was difficult for students to notice that there was anything new to pay attention to here. Recognizing that many students weren't seeing the smaller pieces of cake as objects of a set, Monte concluded that the lack of variety in solutions was most likely linked to the students' rigid conceptualization of fraction as area; therefore, he kept pointing out that "the cake is divided up into much smaller pieces."

Lars and several other students were really anxious to share their drawings, and because Monte hoped that Lars' drawing might help students see the smaller pieces, Monte asked him to share his drawing next. Note how Lars' presentation surfaces a new set of ideas and misconceptions among his classmates. Immediately, most of the students complained that this didn't work, that the shaded region was <u>not</u> ¼ of the cake. When asked to explain why it didn't work, Leticia said that she didn't know, but it just wasn't right. McKenzie, however, thought that she could show that it was ¼ by shading additional similar shapes on Lars' diagram. She came to the whiteboard and added the following:





But then she said, "Oh, I can't make four of them, so it's not ¼. I thought it was, but it's not." Apparently, her inability to find four "cookie cutter" shapes exactly like the one Lars had drawn had convinced her that the shape could not be replicated four times, so Lars' shaded region was not ¼ of the rectangle. McKenzie has surfaced a common misconception that fractions of a region must consist of congruent pieces.

Jacob disagreed with the conclusion that Lars' shaded region was not ¼ of the rectangle. He asked if he could add something to Lars's drawing and then outlined regions on the array to look like this:

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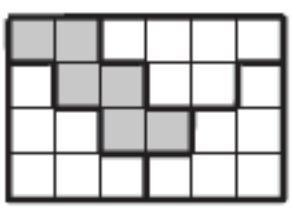
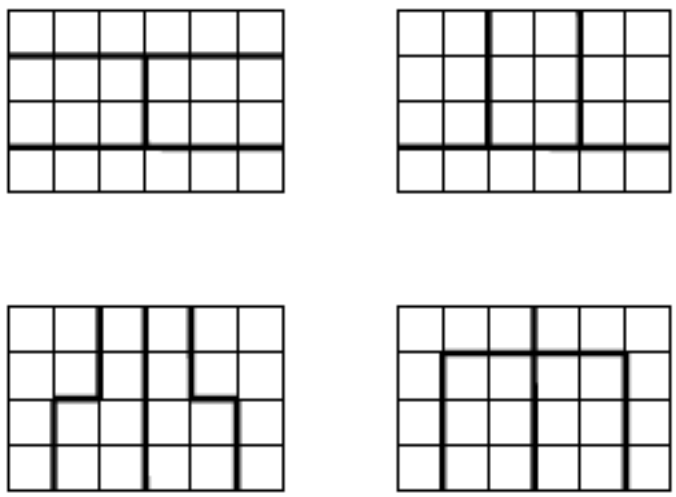


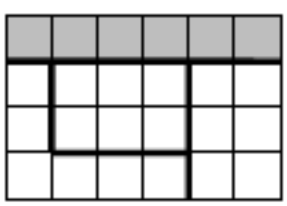
Jacob wrote a big 6 inside each of the four regions, and as he sat down made the claim, "Each of the big portions has 6 pieces, so they are each ¼ of the cake."

Recognizing that Jacob's suggestion could move the class towards seeing fraction as objects within a set, Monte wanted to give students an opportunity to think about Jacob's work. He launched a revised task by posing the following question to the class. "Andrew, Megan and Lars have all suggested different ways that I can eat ¼ of the cake. Will I eat the same amount using each of their strategies?" With this revised task, students will have the opportunity to grapple with the necessity of congruent shapes and contiguous pieces in representing the amount ¼ .

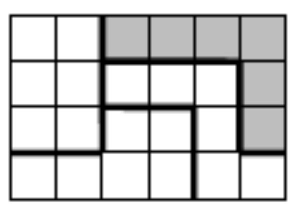
As students thought about this question they realized that each serving contained 6 small pieces of cake, and therefore represented the same amount of cake. Many teachers might assume that students at this point have solidified the idea that if we divide the cake into four portions, each with an equal number of pieces, that each portion is ¼ of the cake. Monte recognizes that this is an emerging idea—somewhat fragile and tenuous—therefore, he allowed students to continue to discuss this idea. It became apparent from student comments that there was much disagreement as to whether this amount should be called ¼ of the cake. At this point, Monte asked students to generate other ways that the cake could be divided into 4 equal servings. As students generated the following diagrams Monte noted that some, but not all students, labeled the 4 portions of the cake as each being ¼.



Most students continued to use some congruent pieces in their diagrams, until Annie and Oskar presented the following:



Annie's drawing



Oskar's drawing

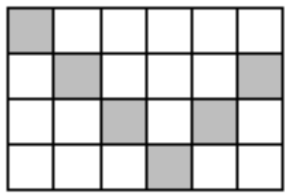
Since Annie and Oskar labeled the shaded portions as ¼ of the cake, Monte asked them how they would convince someone else that the regions they had shaded were actually ¼ of the cake.

Harrison could hardly contain himself as he blurted out, "You have to have four even groups, and all of the groups have to have six pieces in them." (In the moment, Monte assumed that Harrison was using the term "even" to denote "groups of equal size"; however, Monte wondered later if he should have allowed Harrison to clarify his thinking on this point.) Monte decided to pursue Harrison's claim that each portion had to contain six pieces—a claim that had been made much earlier by Jacob, but which seemed to now be more accessible to students. "So what do the rest of you think about Harrison's claim that each group has to have six pieces in them?"

At this point Martin commented, "You can move the 6 shaded pieces in Oskar's drawing to make it look like ¼." Monte let Martin come to the board to show how he would rearrange the six small pieces of cake to form the arrangement that everyone had agreed upon at the beginning of the class period to be ¼ of the cake.

"So, can we call the shaded portion in Oskar's drawing ¼ of the cake?" Monte asked. Most students seemed to be in agreement until Oskar raised another concern. "Yeah, but the pieces have to be connected."

At this point, Monte asked Lars to present another one of his drawings.



Lars' drawing #2

"If I eat Lars' shaded pieces, have I eaten ¼ of the cake?" Monte asked. Monte has carefully orchestrated the discussion to elicit, examine, and begin to dispel two misconceptions that his students had about shading ¼ of an area: that portions need to be congruent and contiguous. Implicit in the work of the students is that the area has to be divided into <u>four</u> portions, and that each portion has to contain the same number of smaller, equal-sized pieces—in this case, six of the twenty-four.

<u>Conclusion</u>

There is still much that needs to be solidified for these students before they fully understand fraction as a portion of a set of objects. For example, will they still see the same relationship if the *pieces* are not *pieces of area,* or when amount means quantity not area? That is, will they see 6 pennies out of 24 as ¼ of the pennies, or 6 students in a class of 24 as ¼ of the class? How will they record the amount using fractions? Will they record $\frac{6}{24}$ and will they recognize that $\frac{6}{24}$ is equivalent to ¼? These are some of the issues that will need to be explored and solidified in future lessons to help these students more fully understand fraction as a portion of a set of objects. And, while all students seemed to follow the flow of ideas as they occurred in this lesson, for at least some of them their understanding is still fragile as evidenced by this summary response from Brooke's paper: ***"1/4 is always one group out of four groups, so there will always be four even, never odd groups."***

As evidenced by this teaching episode, teaching and learning can be very intertwined and interdependent. The CMI Framework provided Monte with principles and tools that helped him understand and exploit this interdependence. Specifically, the framework helped him plan and guide his instruction. It helped him analyze and make sense of his students' work which in turn helped him guide their thinking and focus their learning. Teaching for full and genuine mathematical understanding is challenging. Teachers often need to change their paradigms of mathematics and mathematics teaching, as well as change their instructional practices in order to accomplish this. The CMI Framework provides teachers with key principles, language and a general yet flexible structure to guide and support these changes.