mathematics vision project

Transforming Mathematics Education

Flexible & Engaging // Seamless Common Core Companion

mathematics
vision project

Transforming Mathematics Education

Flexible & Engaging
Seamless Common Core Companion

The Mathematics Vision Project

- Mathematics Vision Project (MVP) Team
- Aligned to Core Standards
- Overview of MVP materials

What is the Mathematics Vision Project?

- The Mathematics Vision Project is an educator-driven initiative to provide first-rate curriculum and professional development resources to secondary math teachers.
- Built from the ground-up by a team of award-winning, math-loving education experts, the material is flexible, engaging, and a seamless companion to the Common Core.
- **We believe math is engaging.** Our desire is to enable educators to teach their students through accessible, engaging, supportive curriculum.

What sets us apart from traditional publishers?

- Our materials are created from the ground-up to embody the focus, coherence, and rigor of the Common Core and immerse students in the Standards for Mathematical Practice.
- Our materials respect what is known about how students learn mathematics as well as the key ideas for how knowledge is organized and generated within the discipline of mathematics.
- Our curriculum is free and published under a Creative Commons license. Other support materials and professional development are available at mathematicsvisionproject.org

Basic Structure of the Materials

- The underlying design of the materials distinguishes between *problems* and *exercises*
- Each problem or exercise has a purpose
  - *teach new knowledge*
  - *bring misconceptions to the surface*
  - *build skill or fluency*
  - *engage students in mathematical practice*
- Assignments are purposefully designed
  - *"Lessons have a few well-designed problems that progressively build and extend understanding."*

(K-8 Publishers' Criteria for the Common Core Standards in Mathematics)

In-Class Tasks and Ready-Set-Go Assignments

- In-class tasks are designed to be facilitated by a teacher, with students working together to build a balance of conceptual understanding and procedural skill.

- Ready-Set-Go assignments are independent practice that reinforce the work done in class and prepare students for upcoming work in class.

For Example:

Module 4 – Linear and Exponential Functions

4.1 Classroom Task: Connecting the Dots: Piggies and Pools – A Develop Understanding Task
Introducing continuous linear and exponential functions **(F.IF.3)**
Ready, Set, Go Homework: Linear and Exponential Functions 4.1

4.2 Classroom Task: Sorting Out the Change – A Solidify Understanding Task
Defining linear and exponential functions based upon the pattern of change (**F.LE.1, F.LE.2**)
Ready, Set, Go Homework: Linear and Exponential Functions 4.2

Problem

Exercise

4.3 Classroom Task: Where's My Change – A Practice Understanding Task
Identifying rates of change in linear and exponential functions (**F.LE.1, F.LE.2**)
Ready, Set, Go Homework: Linear and Exponential Functions 4.3

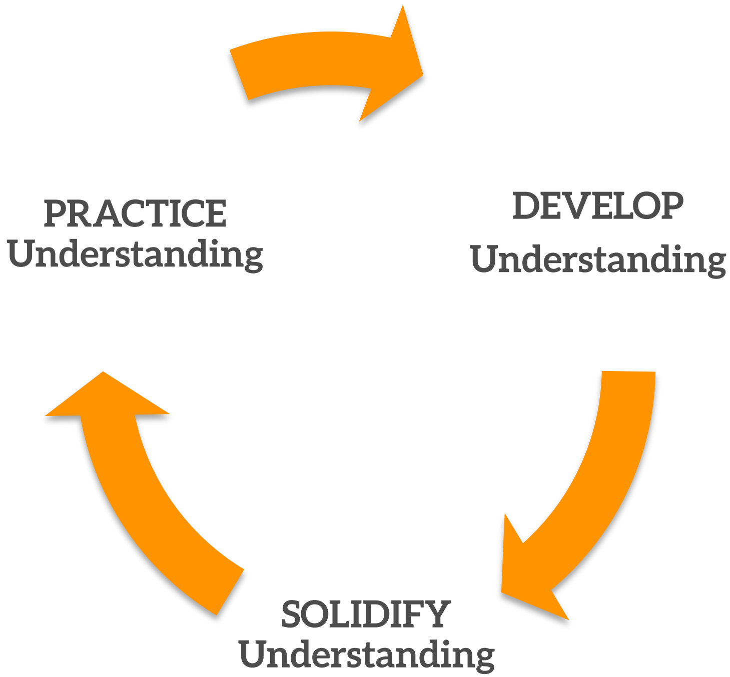
4.4 Classroom Task: Linear, Exponential or Neither – A Practice Understanding Task
Distinguishing between linear and exponential functions using various representations (**F.LE.3, F.LE.5**)
Ready, Set, Go Homework: Linear and Exponential Functions 4.4

4.5 Classroom Task: Getting Down to Business – A Solidify Understanding Task
Comparing the growth of linear and exponential functions (**F.LE.2, F.LE.3, F.LE.5, F.IF.7**)
Ready, Set, Go Homework: Linear and Exponential Functions 4.5

4.6 Classroom Task: Growing, Growing, Gone – A Solidify Understanding Task
Comparing linear and exponential models of population **(F.BF.1, F.BF.2, F.LE.1, F.LE.2, F.LE.3)**
Ready, Set, Go Homework: Linear and Exponential Functions 4.6

Task Sequencing

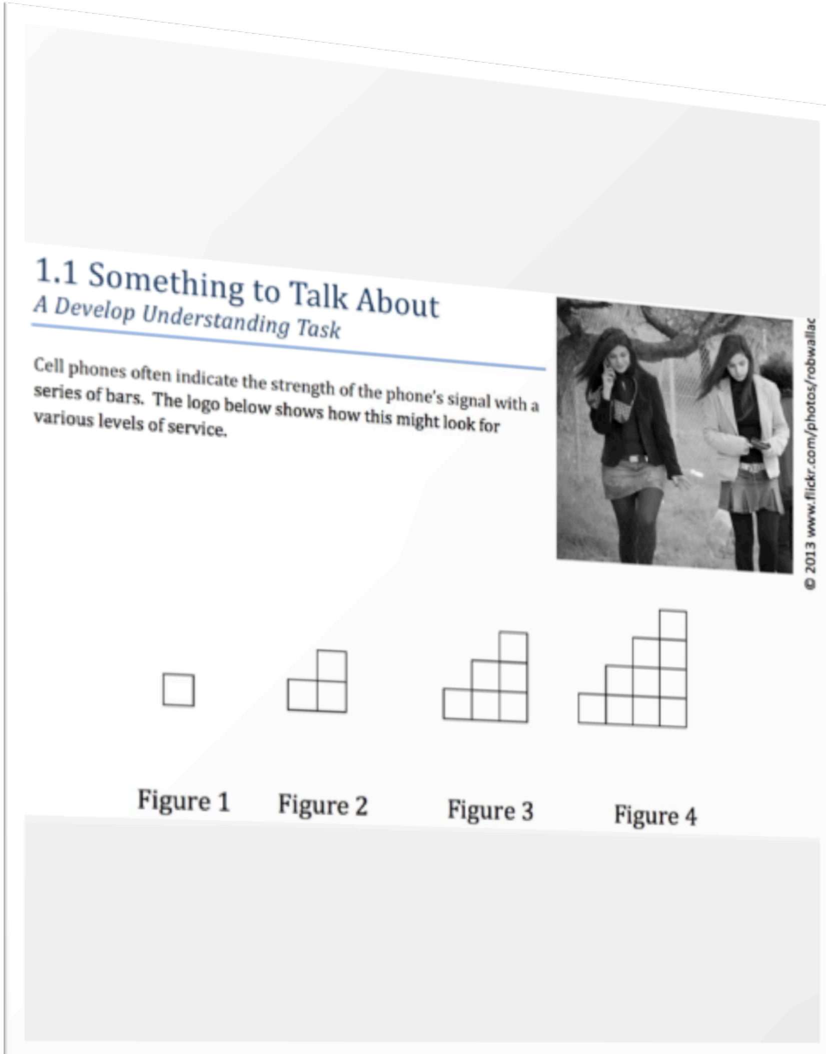
Comprehensive Mathematics Instruction Framework



- *Develop Understanding* tasks surface student thinking
- *Solidify Understanding* tasks examine and extend
- *Practice Understanding* tasks build fluency

Transforming Mathematics Education

mathematics vision project



1.1 Something to Talk About

A Develop Understanding Task

Cell phones often indicate the strength of the phone's signal with a series of bars. The logo below shows how this might look for various levels of service.

Figure 1 Figure 2 Figure 3 Figure 4

Curriculum

In-Class Tasks

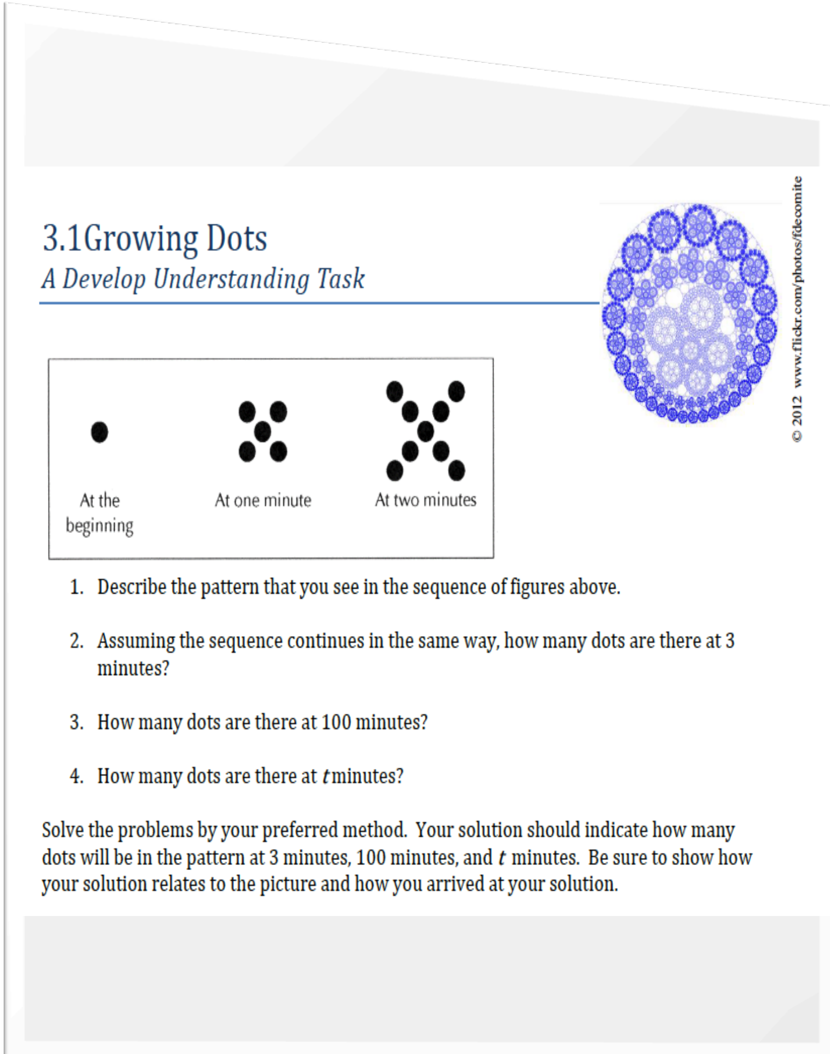
and

Ready, Set, Go! Assignments

Transforming Mathematics Education

mathematics vision project

Growing Dots



3.1Growing Dots

A Develop Understanding Task

© 2012 www.flickr.com/photos/fdecomite

At the beginning | At one minute | At two minutes

1. Describe the pattern that you see in the sequence of figures above.

2. Assuming the sequence continues in the same way, how many dots are there at 3 minutes?

3. How many dots are there at 100 minutes?

4. How many dots are there at t minutes?

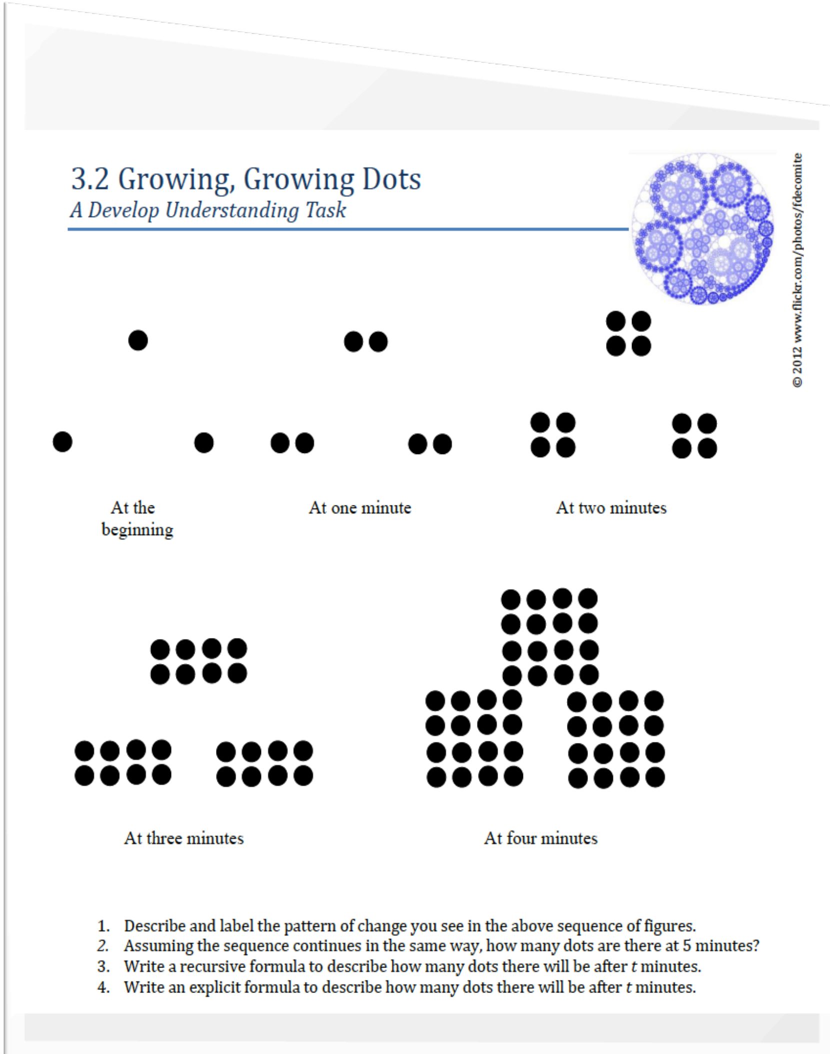
Solve the problems by your preferred method. Your solution should indicate how many dots will be in the pattern at 3 minutes, 100 minutes, and t minutes. Be sure to show how your solution relates to the picture and how you arrived at your solution.

Transforming Mathematics Education

mathematics vision project

Video of Growing Dots

Growing, Growing Dots



3.2 Growing, Growing Dots

A Develop Understanding Task

1. Describe and label the pattern of change you see in the above sequence of figures.
2. Assuming the sequence continues in the same way, how many dots are there at 5 minutes?
3. Write a recursive formula to describe how many dots there will be after *t* minutes.
4. Write an explicit formula to describe how many dots there will be after *t* minutes.

Transforming Mathematics Education

mathematics vision project

What Does It Mean?

3.8 What Does It *Mean*?

A Solidify Understanding Task

Each of the tables below represents an arithmetic sequence. Find the missing terms in the sequence, showing your method.

x	1	2	3
y	5		11

x	1	2	3	4	5
y	18				-10

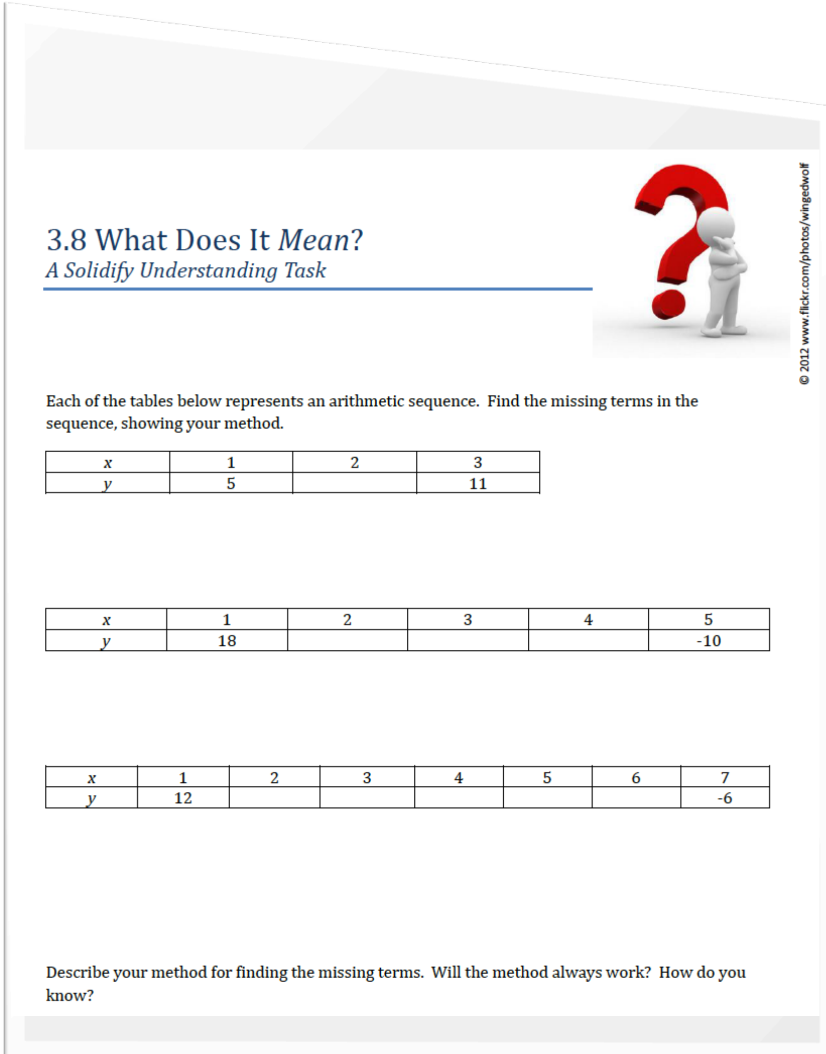
x	1	2	3	4	5	6	7
y	12						-6

Describe your method for finding the missing terms. Will the method always work? How do you know?

Transforming Mathematics Education

mathematics vision project

What Does It Mean?



3.8 What Does It *Mean*?

A Solidify Understanding Task

Each of the tables below represents an arithmetic sequence. Find the missing terms in the sequence, showing your method.

x	1	2	3
y	5		11

x	1	2	3	4	5
y	18				-10

x	1	2	3	4	5	6	7
y	12						-6

Describe your method for finding the missing terms. Will the method always work? How do you know?

$$d = \frac{-10-18}{5-1} \qquad m = \frac{y_2 - y_1}{x_2 - x_1}$$

$$-10 = 18 + 4d$$

$$a_n = a_1 + (n-1)d$$

Transforming Mathematics Education

mathematics vision project

Geometric Meanies

3.9 Geometric *Meanies*

A Solidify and Practice Task

Each of the tables below represents a geometric sequence. Find the missing terms in the sequence, showing your method.

Table 1

x	1	2	3
y	3		12

Is the missing term that you identified the only answer? Why or why not?

Table 2

x	1	2	3	4
y	7			875

Are the missing terms that you identified the only answers? Why or why not?

Table 3

x	1	2	3	4	5
y	6				96

Transforming Mathematics Education

mathematics vision project

3.9 Geometric *Meanies*

A Solidify and Practice Task

Each of the tables below represents a geometric sequence. Find the missing terms in the sequence, showing your method.

Table 1

x	1	2	3
y	3		12

Is the missing term that you identified the only answer? Why or why not?

Table 2

x	1	2	3	4
y	7			875

Are the missing terms that you identified the only answers? Why or why not?

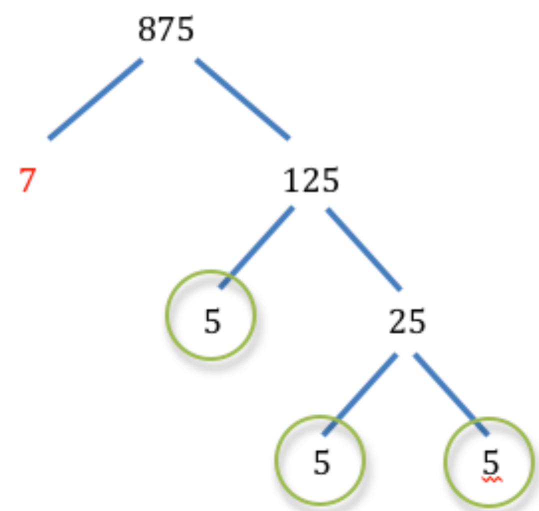
Table 3

x	1	2	3	4	5
y	6				96

Transforming Mathematics Education

Geometric Meanies

$$\sqrt[3]{\frac{875}{7}}$$

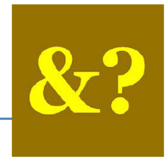


mathematics vision project

I Know, What Do You Know?

3.10 I Know . . . What Do You Know?

A Practice Task



In each of the problems below I share some of the information that I know about a sequence. Your job is to add all the things that you know about the sequence from the information that I have given. Depending on the sequence, some of things you may be able to figure out for the sequence are: a table, a graph, an explicit equation, a recursive equation, the constant ratio or constant difference between consecutive terms, any terms that are missing, the type of sequence, or a story context. Try to find as many as you can for each sequence, but you must have at least 4 things for each.

1. I know that: the recursive formula for the sequence is $f(1) = -12,\ f(n) = f(n-1) + 4$
   What do you know?

2. I know that: the first 5 terms of the sequence are 0, -6, -12, -18, -25 . . .
   What do you know?

3. I know that: the explicit formula for the sequence is $f(n) = -10(3)^n$
   What do you know?

Transforming Mathematics Education

mathematics vision project

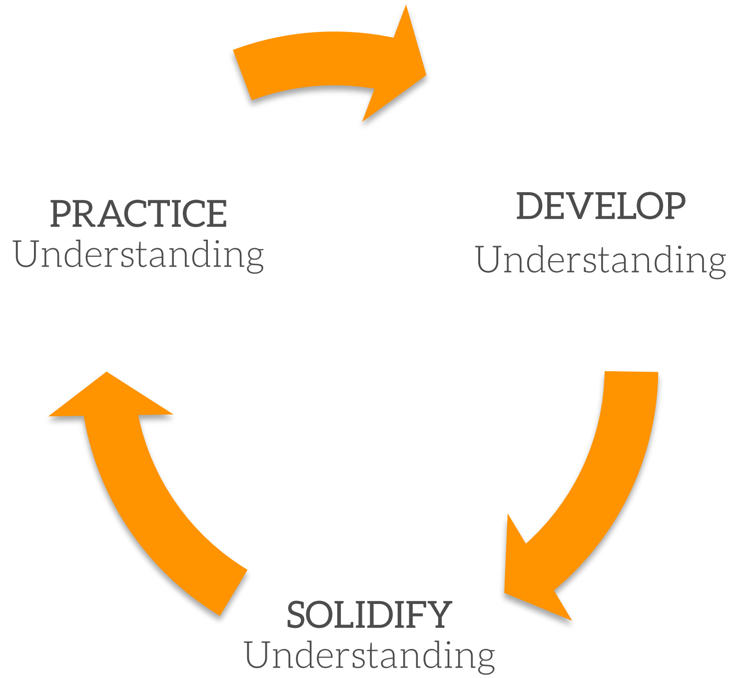
Using Representations

- What do we know about linear and exponential functions based on these tasks?
- How do these ideas emerge from the representations?

Take a Look at the Ready-Set-Go Homework

Curriculum Design Principles Supported by the Learning Cycle

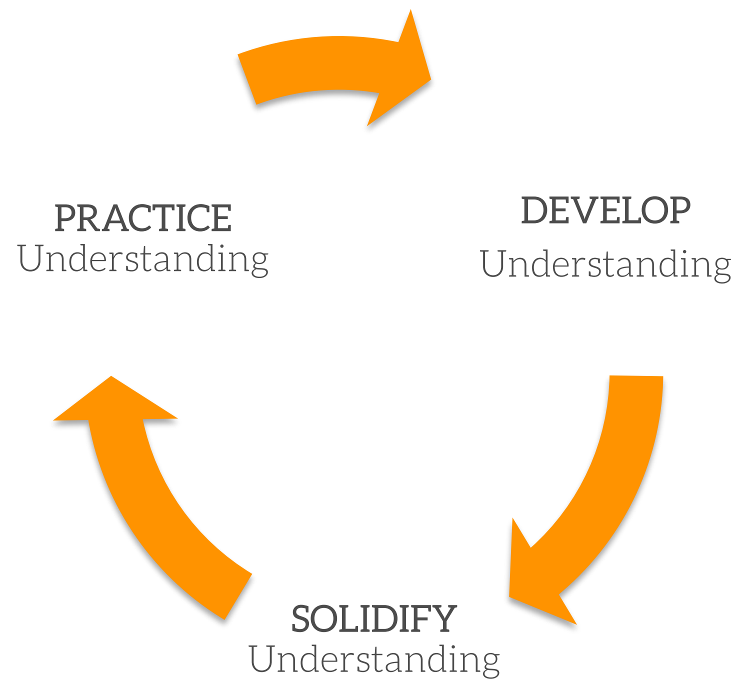
- What constitutes a Worthwhile Mathematical Task?
- Contextualized or abstract/ symbolic?
- Amount of scaffolding provided?
- Constraints?
- Level of cognitive demand?



Curriculum Design Principles Supported by the Learning Cycle

Thinking Through a Unit

- Ideas: What do we want students to know?
- Strategies: What do we want students to be able to do?
- Representations: How do we want students to make their thinking visible?
- How do ideas, strategies and representations differ from the beginning to the end of a unit of instruction?



How does the experience of the learner change throughout the learning cycle?

Information Transmission Model

Show and Tell
- *Teacher's Role:*
- Provide examples, definitions and properties, procedures, and models

Practice
- Students' Role
- Replicate, drill, practice, and memorize

Comprehensive Mathematics Instruction Model

Problem Solving, Reasoning and Proving, Modeling

Develop
- *Teacher's Role*: Provide experiences, orchestrate discussions using the 5 practices
- *Students' Role*: Notice patterns; make conjectures; create arguments

Solidify
- *Teacher's Role*: Provide experiences, orchestrate discussions using the 5 practices
- *Students' Role*: See structure; see regularities; attend to precision; create and critique

Practice
- *Teacher's Role*: Provide a vehicle for practice, provide feedback
- *Students' Role*: Reason quantitatively; work towards efficiency, flexibility, accuracy; Apply (model with mathematics)

Develop Understanding Tasks

- Low threshold, high ceiling (easy entry, but extendable for all learners)
- Contextualized (problematic story context, diagrams, symbols)
- Multiple pathways to solutions or multiple solutions
- Surface student thinking (misconceptions and correct thinking)
- Purposeful selection of the vocabulary, numbers, etc. to reveal rather than obscure the mathematics
- Introduction of a number of representations

Solidify Understanding Tasks

Features of the task (context, scaffolding questions, constraints) focuses students' attention on:

- looking for patterns and making use of structure
- looking for repeated reasoning and expressing regularities as generalized methods
- attending to precision in language and use of symbols
- constructing viable arguments and critiquing the reasoning of others
- using representations and tools strategically for the purpose of developing deeper levels of understanding of mathematical ideas, strategies, and/or representations

Practice Understanding Tasks

Practice tasks focused on acquiring fluency:

- Task involves either reproducing previously learned facts, definitions, rules, formulas or models; OR drawing upon previously learned facts, definitions, rules, formulas or models; OR committing facts, definitions, rules, formulas or models to memory
- An appropriate vehicle of practice is selected (e.g., routines, games, worksheets, etc.) which allows for reproducing, drawing upon, or committing to memory previously examined mathematics
- Task focuses on a broad definition of fluency: accuracy, efficiency, flexibility, automaticity

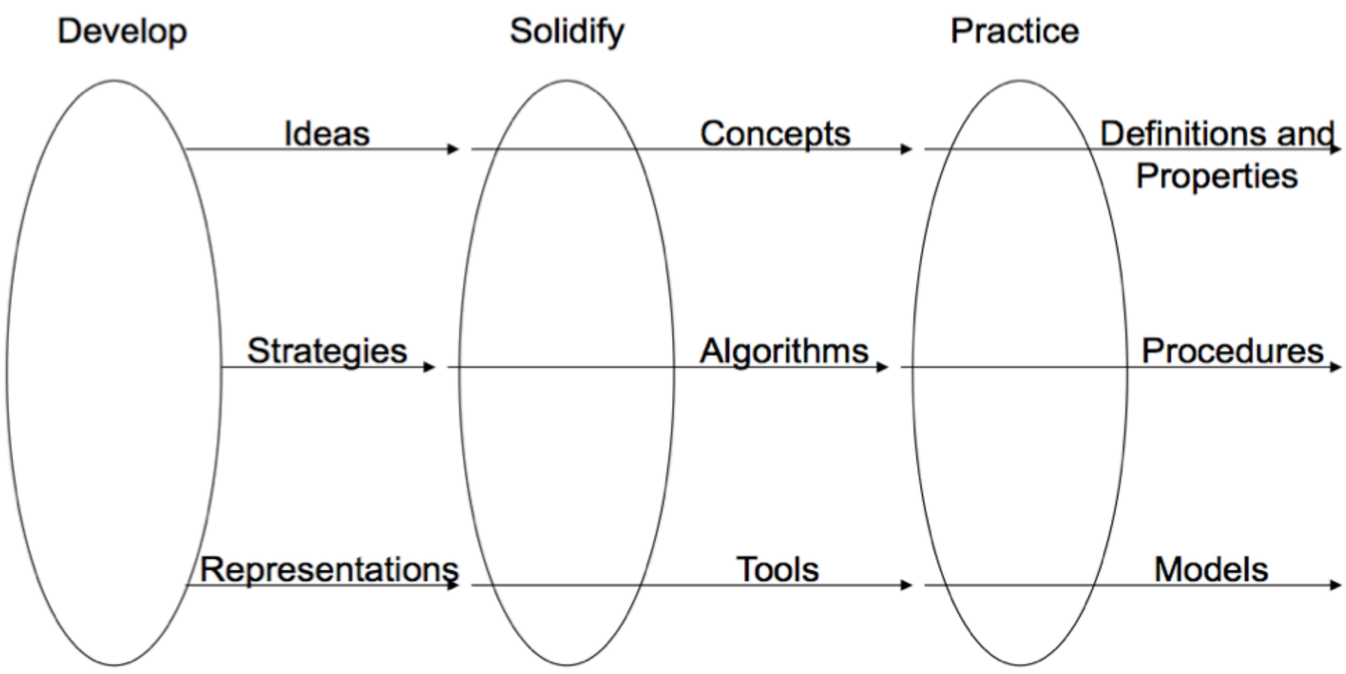
Practice Understanding Tasks

Practice tasks focused on refining understanding:

- Task allows student to use reasoning habits to contextualize (symbolic to real-world) and decontextualize (real-world to symbolic) problems and situations.
- Tasks involve sufficient complexity to refine mathematical thinking beyond rote memorization
- The task requires a high level of cognitive demand because students are required to draw upon multiple concepts and procedures, make use of structure and recognize complex relationships among facts, definitions, rules, formulas and/ or models

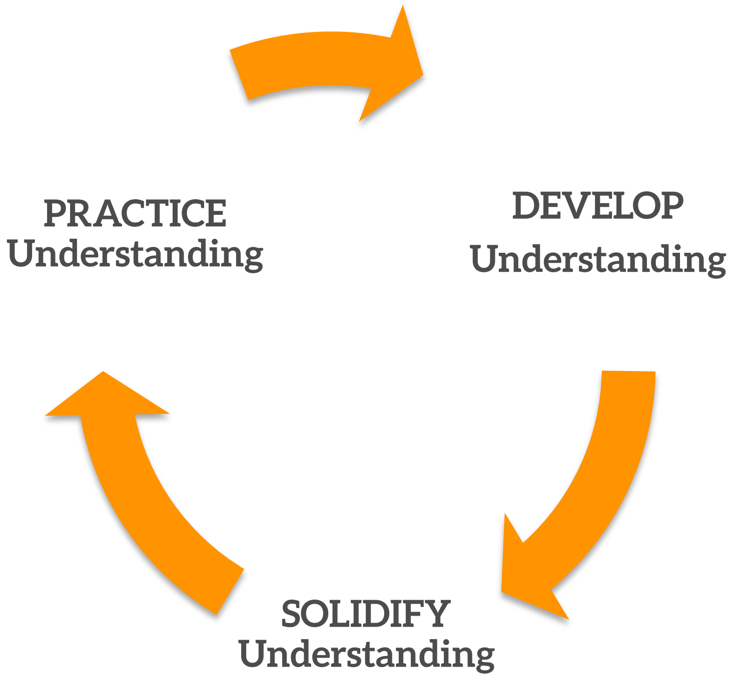
Unpacking the Mathematics of the Learning Cycle

- What is the conceptual, procedural, and representational understanding that is emerging from the work in the learning cycle?
- How does the mathematical understanding change from the beginning to the end of the learning cycle?



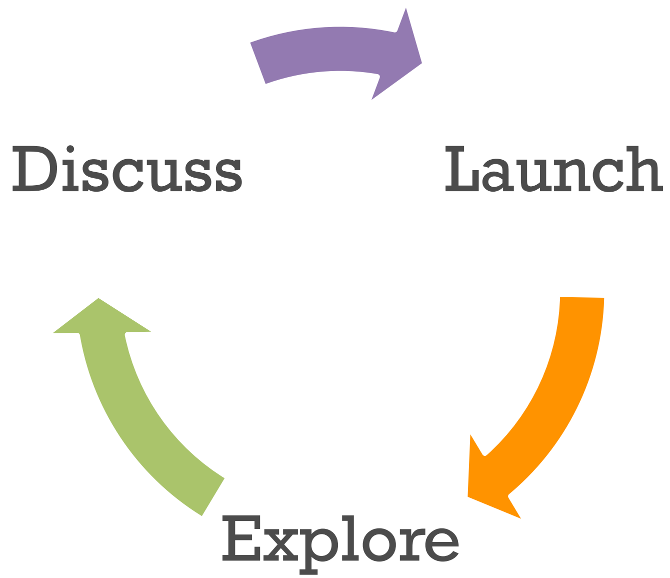
Task Sequencing

Comprehensive Mathematics Instruction Framework



- *Develop Understanding* tasks surface student thinking
- *Solidify Understanding* tasks examine and extend
- *Practice Understanding* tasks build fluency

The *Teaching Cycle*



The *Teaching Cycle*: Launch

How will you . . .

- hook and motivate students;
- provide schema (the problem setting, the mathematical context, and the challenge) for the mathematical task;
- provide tools, information, vocabulary, conventions and notations, as necessary; and
- describe what the expectations are for the finished task without giving away too much of the problem and leaving the potential of the task intact?

The *Teaching Cycle*: Explore

- How will you organize and encourage students to explore, investigate, experiment, look for patterns, make conjectures, collect and record data, participate in group discussions, and revisit and revise their thinking relative to the mathematical ideas intended to be elicited by the task?

- What will you look for and listen for as you observe students?

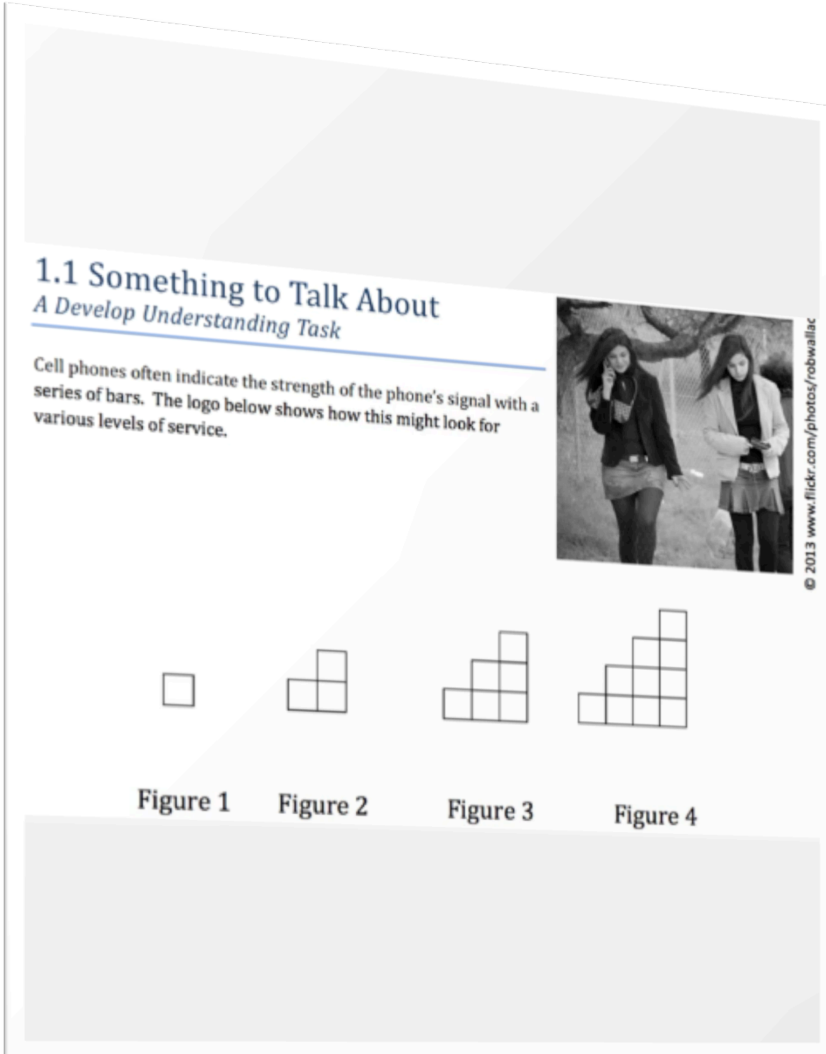
- What will you accept as evidence of student understanding?

- What questions will you ask to stimulate, redirect, focus, and extend the students' mathematical thinking?

The *Teaching Cycle*: Discuss

- How will you select which students will present and discuss their solutions and strategies?
- How will you determine what ideas to pursue in depth and what to defer for another time?
- How will you decide whether to contribute to the discourse by providing additional information (e.g., vocabulary, conventions, notation), suggesting other models, demonstrating alternative strategies, clarifying difficult issues; or to allow students to continue to struggle to make sense of an idea or concept?

Something to Talk About


1.1 Something to Talk About
A Develop Understanding Task

Cell phones often indicate the strength of the phone's signal with a series of bars. The logo below shows how this might look for various levels of service.

© 2013 www.flickr.com/photos/robwallac

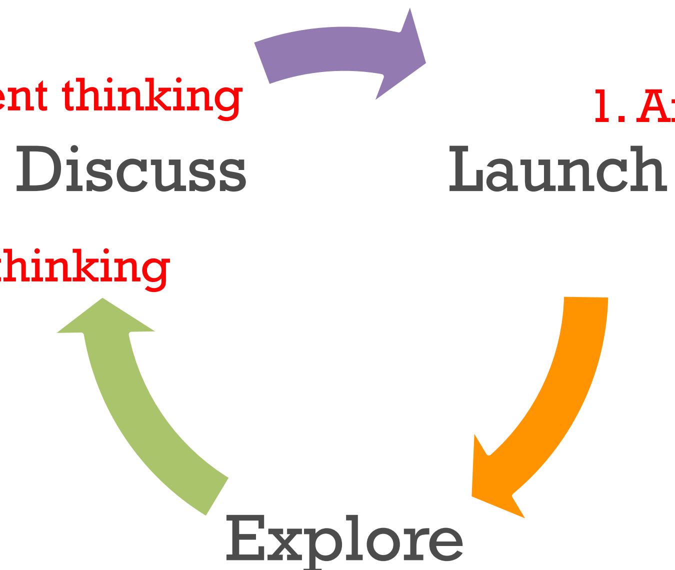
Figure 1 Figure 2 Figure 3 Figure 4

Transforming Mathematics Education

mathematics vision project

The *Teaching Cycle*

Connected to the 5 practices of Orchestrating Discussions



1. Anticipate student thinking
2. Monitor student thinking
3. Select student thinking
4. Sequence student thinking
5. Connect student thinking

Five Practices for Facilitating Mathematical Discourse

1. **Anticipate** student thinking. Work the task yourself.

2. **Monitor** students as they work. Circulate around the room and ask students how they are thinking about what they have written.

3. **Select** students for the classroom discussion. Have a method for keeping track of who you have selected
4. **Sequence** student work.

5. **Connect.** Help students to make connections among the ideas presented.

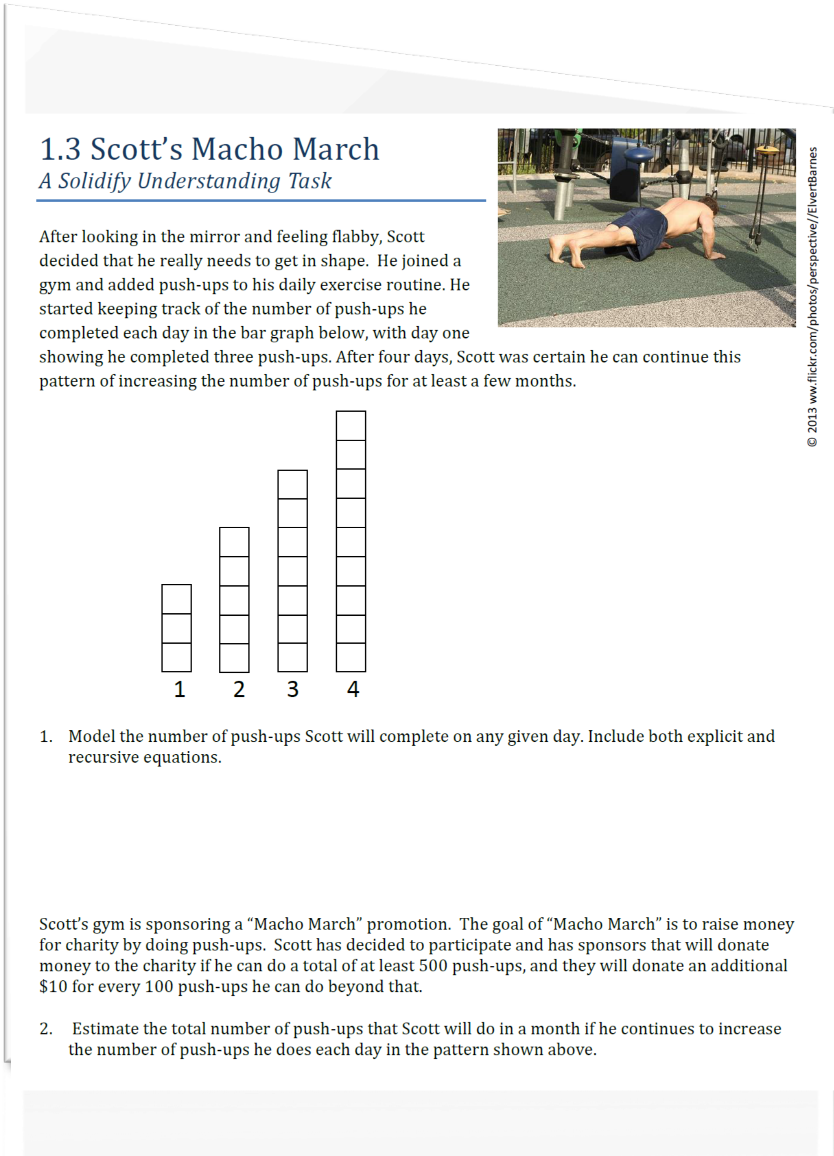
The 0 Practice

Have clear mathematical goals and a task that supports the goals.

Something to Talk About

- How is the rate of growth for this context different from the rate of growth in *Growing Dots* and *Growing, Growing Dots*?
- How does the rate of growth show up in different representations? (e.g., in a table, in a graph, in a recursive rule)
- How can we represent this sequence with an explicit equation?
- How does the structure of the diagram support this work?

Scott's Macho March


1.3 Scott's Macho March

A Solidify Understanding Task

After looking in the mirror and feeling flabby, Scott decided that he really needs to get in shape. He joined a gym and added push-ups to his daily exercise routine. He started keeping track of the number of push-ups he completed each day in the bar graph below, with day one showing he completed three push-ups. After four days, Scott was certain he can continue this pattern of increasing the number of push-ups for at least a few months.

1. Model the number of push-ups Scott will complete on any given day. Include both explicit and recursive equations.

Scott's gym is sponsoring a "Macho March" promotion. The goal of "Macho March" is to raise money for charity by doing push-ups. Scott has decided to participate and has sponsors that will donate money to the charity if he can do a total of at least 500 push-ups, and they will donate an additional $10 for every 100 push-ups he can do beyond that.

2. Estimate the total number of push-ups that Scott will do in a month if he continues to increase the number of push-ups he does each day in the pattern shown above.

Transforming Mathematics Education

mathematics vision project

Scott's Macho March

- How did you extend your thinking that emerged in *Something to Talk About* as you worked on this task?
- What ideas, strategies or representations got solidified for you?
- What connections can be made between the work of the first part of the task (describing push ups per day) and the last part of the task (describing the accumulated push ups for the month)? Why do these relationships exist?

1.6 The Tortoise and the Hare

A Solidify Understanding Task

In the children's story of the tortoise and the hare, the hare mocks the tortoise for being slow. The tortoise replies, "Slow and steady wins the race." The hare says, "We'll just see about that," and challenges the tortoise to a race. The distance from the starting line of the hare is given by the function:

$d = t^2$ (d in meters and t in seconds)

Because the hare is so confident that he can beat the tortoise, he gives the tortoise a 1 meter head start. The distance from the starting line of the tortoise including the head start is given by the function:

$d = 2^t$ (d in meters and t in seconds)

1. At what time does the hare catch up to the tortoise?

2. If the race course is very long, who wins: the tortoise or the hare? Why?

3. At what time(s) are they tied?

The Tortoise and the Hare

Transforming Mathematics Education

mathematics vision project

Tortoise and the Hare

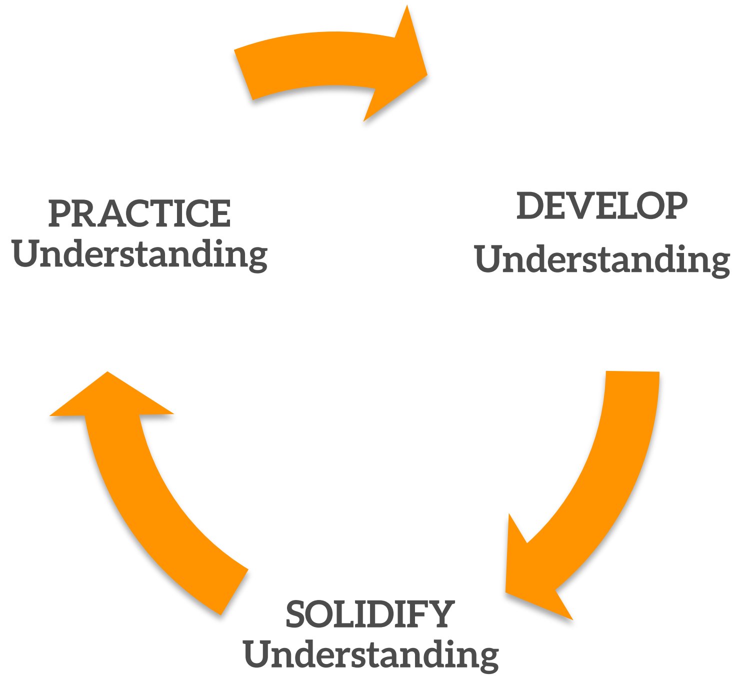
- How are the contexts of the three tasks worked on today similar, and how are they different?
- How are the quadratic representations we have seen today similar and how are they different?
- How does quadratic and exponential behavior compare?

What Are the Features of Quadratic Functions?

- Based on your work in these three tasks, what are some of the defining features of quadratic functions?
- More generally what ideas about function can we add to the work form Secondary Math I ?
- To this point, what similarities and differences do you notice in the nature of the work from Math I and Math II?

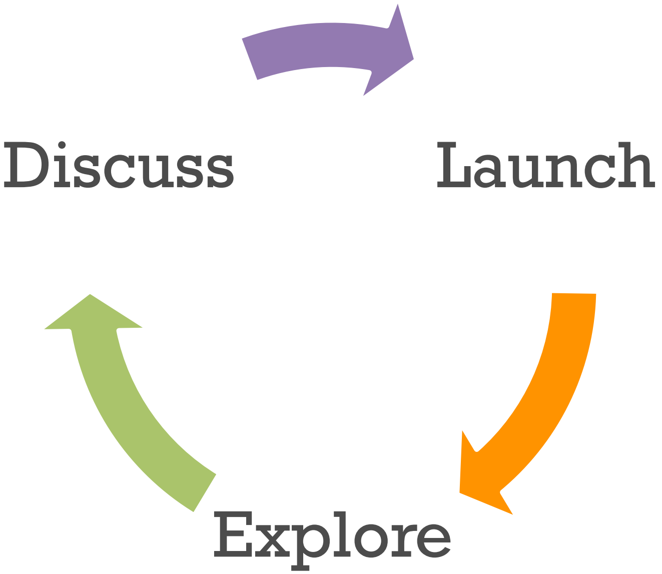
Day 2

Task Sequencing
Comprehensive Mathematics Instruction Framework



- *Develop Understanding* tasks surface student thinking
- *Solidify Understanding* tasks examine and extend
- *Practice Understanding* tasks build fluency

The *Teaching Cycle*



The *Teaching Cycle*: Launch

How will you . . .

- hook and motivate students;
- provide schema (the problem setting, the mathematical context, and the challenge) for the mathematical task;
- provide tools, information, vocabulary, conventions and notations, as necessary; and
- describe what the expectations are for the finished task without giving away too much of the problem and leaving the potential of the task intact?

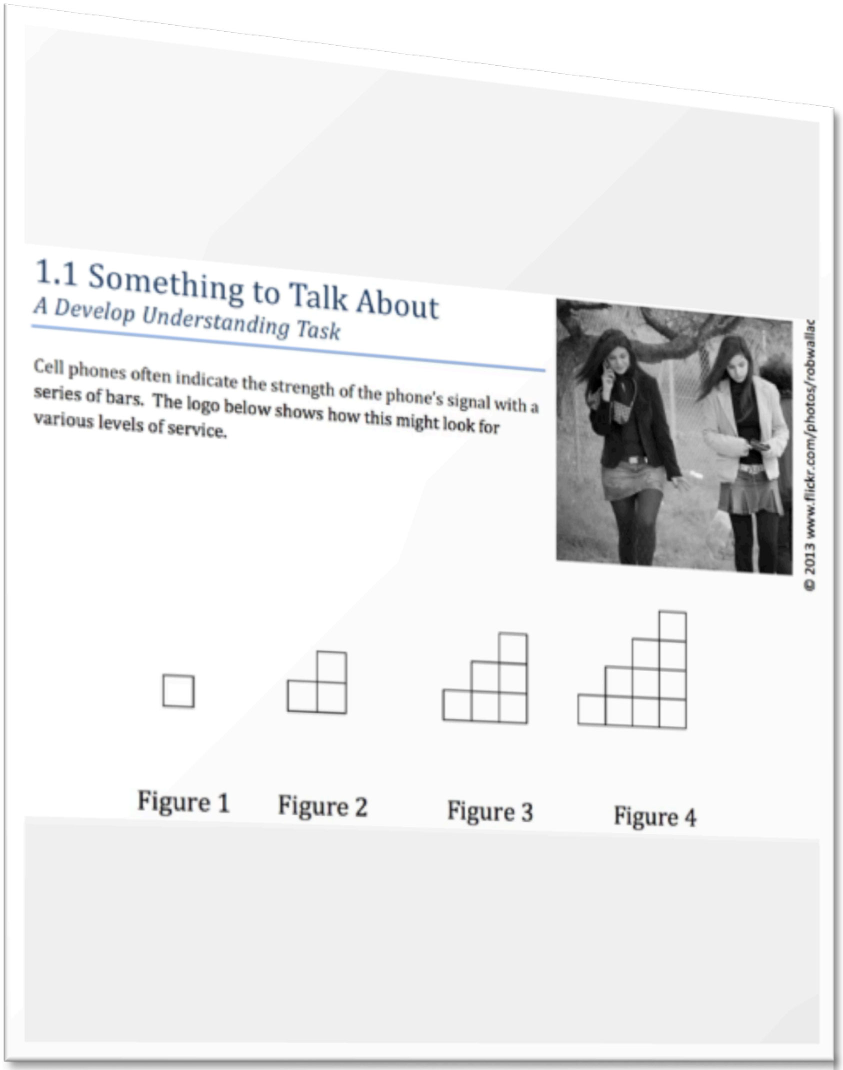
The *Teaching Cycle*: Explore

- How will you organize and encourage students to explore, investigate, experiment, look for patterns, make conjectures, collect and record data, participate in group discussions, and revisit and revise their thinking relative to the mathematical ideas intended to be elicited by the task?
- What will you look for and listen for as you observe students?
- What will you accept as evidence of student understanding?
- What questions will you ask to stimulate, redirect, focus, and extend the students' mathematical thinking?

The *Teaching Cycle*: Discuss

- How will you select which students will present and discuss their solutions and strategies?
- How will you determine what ideas to pursue in depth and what to defer for another time?
- How will you decide whether to contribute to the discourse by providing additional information (e.g., vocabulary, conventions, notation), suggesting other models, demonstrating alternative strategies, clarifying difficult issues; or to allow students to continue to struggle to make sense of an idea or concept?

Something to Talk About



1.1 Something to Talk About
A Develop Understanding Task

Cell phones often indicate the strength of the phone's signal with a series of bars. The logo below shows how this might look for various levels of service.

© 2013 www.flickr.com/photos/robwallac

Figure 1 Figure 2 Figure 3 Figure 4

Transforming Mathematics Education

mathematics vision project

The *Teaching Cycle*

Connected to the 5 practices of Orchestrating Discussions

5. Connect student thinking

Discuss

Launch

1. Anticipate student thinking

4. Sequence student thinking

Explore

2. Monitor student thinking

3. Select student thinking

Five Practices for Facilitating Mathematical Discourse

1. **Anticipate** student thinking. Work the task yourself.

2. **Monitor** students as they work. Circulate around the room and ask students how they are thinking about what they have written.

3. **Select** students for the classroom discussion. Have a method for keeping track of who you have selected
4. **Sequence** student work.

5. **Connect.** Help students to make connections among the ideas presented.

The 0 Practice

Have clear mathematical goals and a task that supports the goals.

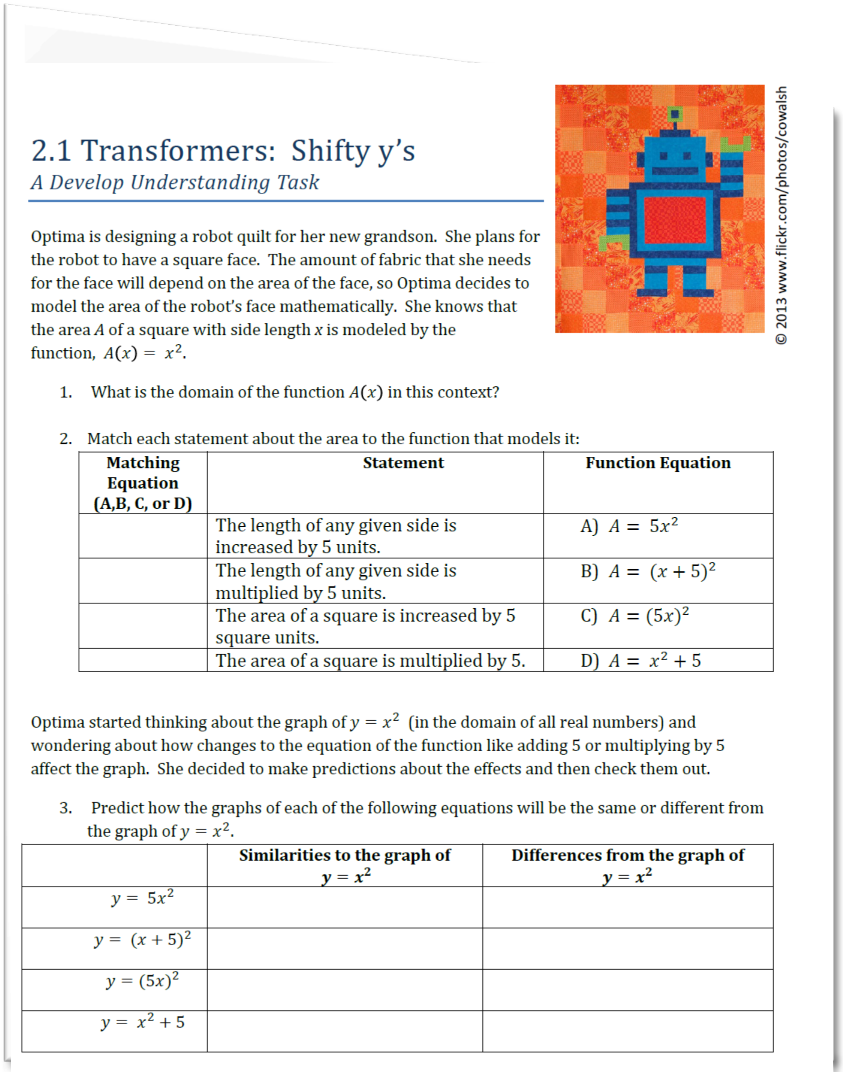
Something to Talk About

- How is the rate of growth for this context different from the rate of growth in *Growing Dots* and *Growing, Growing Dots*?
- How does the rate of growth show up in different representations? (e.g., in a table, in a graph, in a recursive rule)
- How can we represent this sequence with an explicit equation?
- How does the structure of the diagram support this work?

What Are the Features of Quadratic Functions?

- Based on your work in these tasks, what are some of the defining features of quadratic functions?
- More generally what ideas about function can we add to the work form Secondary Math I ?
- To this point, what similarities and differences do you notice in the nature of the work from Math I and Math II?

Transformers: Shifty y's



2.1 Transformers: Shifty y's

A Develop Understanding Task

Optima is designing a robot quilt for her new grandson. She plans for the robot to have a square face. The amount of fabric that she needs for the face will depend on the area of the face, so Optima decides to model the area of the robot's face mathematically. She knows that the area A of a square with side length x is modeled by the function, $A(x) = x^2$.

1. What is the domain of the function $A(x)$ in this context?

2. Match each statement about the area to the function that models it:

Matching Equation (A,B, C, or D)	Statement	Function Equation
	The length of any given side is increased by 5 units.	A) $A = 5x^2$
	The length of any given side is multiplied by 5 units.	B) $A = (x+5)^2$
	The area of a square is increased by 5 square units.	C) $A = (5x)^2$
	The area of a square is multiplied by 5.	D) $A = x^2 + 5$

Optima started thinking about the graph of $y = x^2$ (in the domain of all real numbers) and wondering about how changes to the equation of the function like adding 5 or multiplying by 5 affect the graph. She decided to make predictions about the effects and then check them out.

3. Predict how the graphs of each of the following equations will be the same or different from the graph of $y = x^2$.

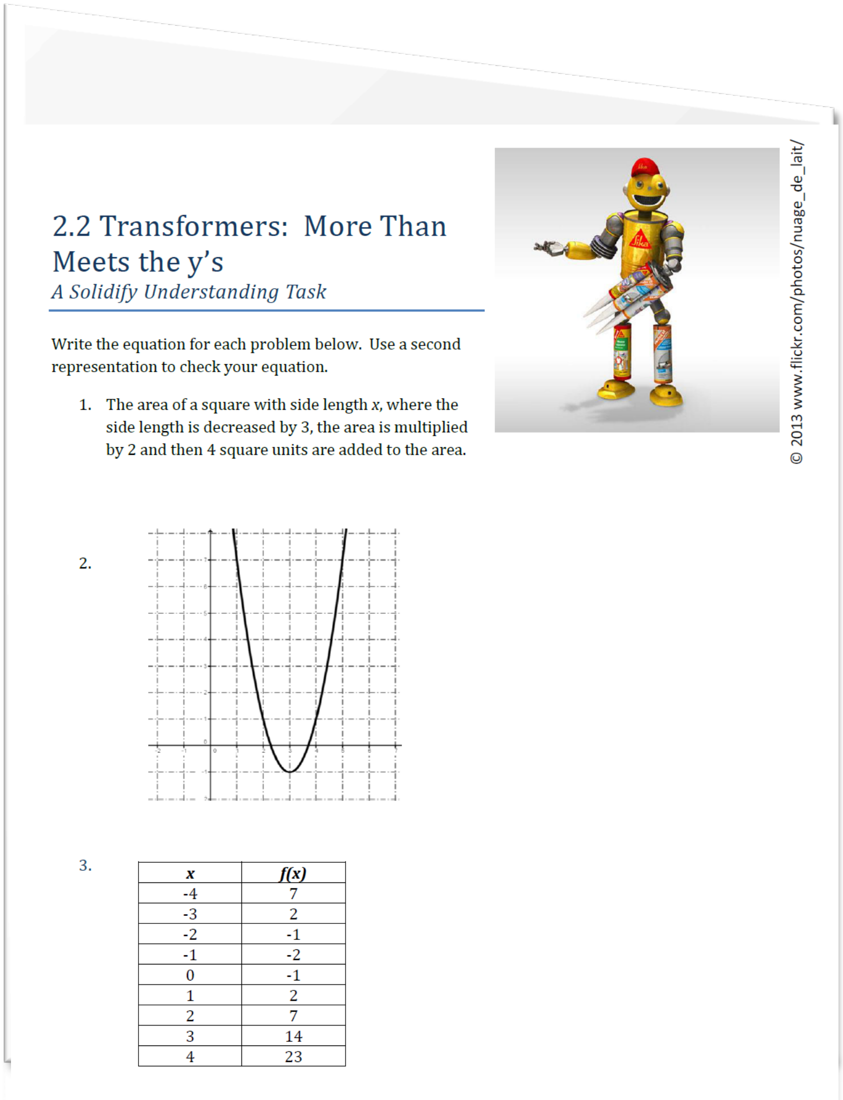
	Similarities to the graph of $y = x^2$	Differences from the graph of $y = x^2$
$y = 5x^2$		
$y = (x+5)^2$		
$y = (5x)^2$		
$y = x^2 + 5$		

Transforming Mathematics Education

mathematics vision project

Transformers: Shifty y's

- What is the goal of this task?
- What ideas are available for students after doing this task?



2.2 Transformers: More Than Meets the y's

A Solidify Understanding Task

Write the equation for each problem below. Use a second representation to check your equation.

1. The area of a square with side length *x*, where the side length is decreased by 3, the area is multiplied by 2 and then 4 square units are added to the area.

2.

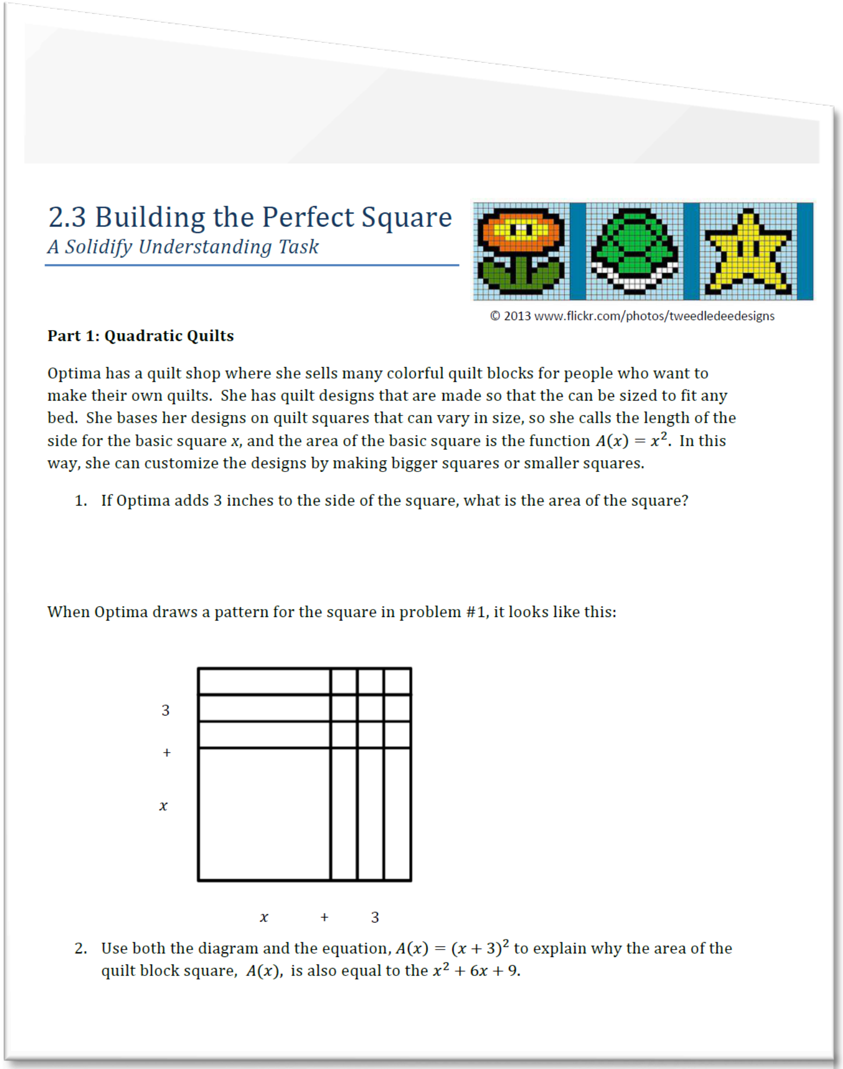
3.

x	*f(x)*
-4	7
-3	2
-2	-1
-1	-2
0	-1
1	2
2	7
3	14
4	23

Transformers: More Than Meets the y's

Transforming Mathematics Education

mathematics vision project



2.3 Building the Perfect Square

A Solidify Understanding Task

© 2013 www.flickr.com/photos/tweedledeedesigns

Part 1: Quadratic Quilts

Optima has a quilt shop where she sells many colorful quilt blocks for people who want to make their own quilts. She has quilt designs that are made so that the can be sized to fit any bed. She bases her designs on quilt squares that can vary in size, so she calls the length of the side for the basic square x, and the area of the basic square is the function $A(x) = x^2$. In this way, she can customize the designs by making bigger squares or smaller squares.

1. If Optima adds 3 inches to the side of the square, what is the area of the square?

When Optima draws a pattern for the square in problem #1, it looks like this:

3 + x

x + 3

2. Use both the diagram and the equation, $A(x) = (x + 3)^2$ to explain why the area of the quilt block square, $A(x)$, is also equal to the $x^2 + 6x + 9$.

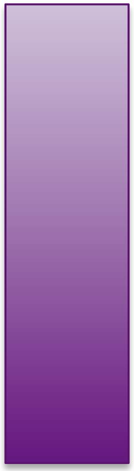
Building the Perfect Square

Transforming Mathematics Education

mathematics vision project

Putting Pieces Together

Square
side length = x units
Area = x^2 square units



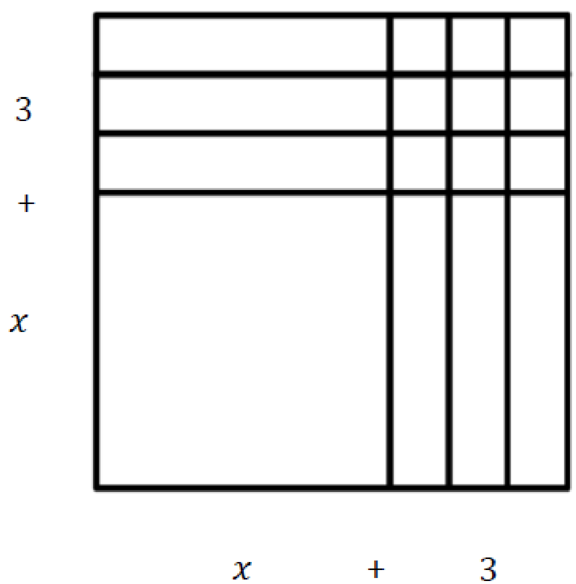
Rectangle
length = x units
width = 1 unit
Area = x square units

Square
side length = 1 unit
Area = 1 square unit

Optima has a quilt shop where she sells many colorful quilt blocks for people who want to make their own quilts. She has quilt designs that are made so that the can be sized to fit any bed. She bases her designs on quilt squares that can vary in size, so she calls the length of the side for the basic square x, and the area of the basic square is the function $A(x) = x^2$. In this way, she can customize the designs by making bigger squares or smaller squares.

1. If Optima adds 3 inches to the side of the square, what is the area of the square?

When Optima draws a pattern for the square in problem #1, it looks like this:



Building the Perfect Square

- What is the connection between completing the square and transformations?

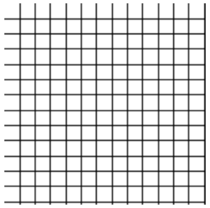
Throwing an Interception

3.5 Throwing an Interception

A Develop Understanding Task

The x-intercept(s) of the graph of a function $f(x)$ are often very important because they are the solution to the equation $f(x)=0$. In past tasks, we learned how to find the x-intercepts of the function by factoring, which works great for some functions, but not for others. In this task we are going to work on a process to find the x-intercepts of any quadratic function that has them. We'll start by thinking about what we already know about a few specific quadratic functions and then use what we know to generalize to all quadratic functions with x-intercepts.

1. What can you say about the graph of the function $f(x)=x^2-2x-3$?
    a. Graph the function

    b. What is the equation of the line of symmetry?
    c. What is the vertex of the function?

2. Now let's think specifically about the x-intercepts.
    a. What are the x-intercepts of $f(x)=x^2-2x-3$?
    b. How far are the x-intercepts from the line of symmetry?
    c. If you knew the line of symmetry was the line $x = h$, and you know how far the x-intercepts are from the line of symmetry, how would you find the actual x-intercepts?
    d. How far above the vertex are the x-intercepts?
    e. What is the value of $f(x)$ at the x-intercepts?

Transforming Mathematics Education

mathematics vision project

Throwing an Interception

- What is the main focus of this task?
- How would you describe the approach to finding the x-intercepts?
- How is the need for the quadratic formula motivated?

Building Procedural Fluency from Conceptual Understanding

What are the foundational concepts for the procedures in these tasks?

- Graphing parabolas using transformations
- Completing the square
- Quadratic formula

Building Procedural Fluency from Conceptual Understanding

"Effective teaching of mathematics builds fluency with procedures on a foundation of conceptual understanding so that students, over time, become skillful in using procedures flexibly as they solve contextual and mathematical problems."

Principles to Actions: Ensuring Mathematical Success for All,
National Council Teachers of Mathematics 2014

Building Procedural Fluency from Conceptual Understanding

- How does the conceptual approach based on students' knowledge of functions provide access to the procedures?

- How does the conceptual approach based on students' knowledge of functions provide motivation for the procedures?

Brutus Bites Back

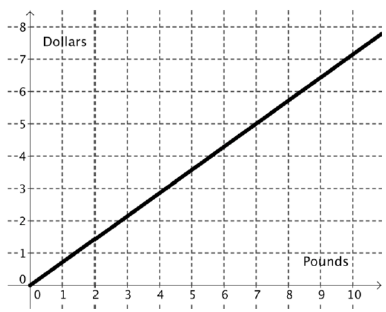
1.1 Brutus Bites Back

A Develop Understanding Task

Remember Carlos and Clarita? A couple of years ago, they started earning money by taking care of pets while their owners are away. Due to their amazing mathematical analysis and their loving care of the cats and dogs that they take in, Carlos and Clarita have made their business very successful. To keep the hungry dogs fed, they must regularly buy Brutus Bites, the favorite food of all the dogs.

Carlos and Clarita have been searching for a new dog food supplier and have identified two possibilities. The Canine Catering Company, located in their town, sells 7 pounds of food for $5.

Carlos thought about how much they would pay for a given amount of food and drew this graph:



1. Write the equation of the function that Carlos graphed.

Transforming Mathematics Education

mathematics vision project

Brutus Bites Back

- What conceptual understandings about inverses are surfaced in this task?

1.2 Flipping Ferraris

A Solidify Understanding Task

© 2013 www.flickr.com/photos/98octane

When people first learn to drive, they are often told that the faster they are driving, the longer it will take to stop. So, when you're driving on the freeway, you should leave more space between your car and the car in front of you than when you are driving slowly through a neighborhood. Have you ever wondered about the relationship between how fast you are driving and how far you travel before you stop, after hitting the brakes?

1. Think about it for a minute. What factors do you think might make a difference in how far a car travels after hitting the brakes?

There has actually been quite a bit of experimental work done (mostly by police departments and insurance companies) to be able to mathematically model the relationship between the speed of a car and the braking distance (how far the car goes until it stops after the driver hits the brakes).

2. Imagine your dream car. Maybe it is a Ferrari 550 Maranello, a super-fast Italian car. Experiments have shown that on smooth, dry roads, the relationship between the braking distance (d) and speed (s) is given by $d(s) = 0.03s^2$. Speed is given in miles/hour and the distance is in feet.
   a) How many feet should you leave between you and the car in front of you if you are driving the Ferrari at 55 mi/hr?
   b) What distance should you keep between you and the car in front of you if you are driving at 100 mi/hr?
   c) If an average car is about 16 feet long, about how many car lengths should you have between you and that car in front of you if you are driving 100 mi/hr?

Flipping Ferraris

Transforming Mathematics Education

mathematics vision project

Flipping Ferraris

- What conceptual understandings about inverses are solidified in this task?
- What procedural ideas are being developed?

Tracking the Tortoise

1.3 Tracking the Tortoise

A Solidify Understanding Task

You may remember a task from last year about the famous race between the tortoise and the hare. In the children's story of the tortoise and the hare, the hare mocks the tortoise for being slow. The tortoise replies, "Slow and steady wins the race." The hare says, "We'll just see about that," and challenges the tortoise to a race.

In the task, we modeled the distance from the starting line that both the tortoise and the hare travelled during the race. Today we will consider only the journey of the tortoise in the race.

Because the hare is so confident that he can beat the tortoise, he gives the tortoise a 1 meter head start. The distance from the starting line of the tortoise including the head start is given by the function:

$d(t) = 2^t$ (d in meters and t in seconds)

The tortoise family decides to watch the race from the sidelines so that they can see their darling tortoise sister, Shellie, prove the value of persistence.

1. How far away from the starting line must the family be, to be located in the right place for Shellie to run by 10 seconds after the beginning of the race? After 20 seconds?

2. Describe the graph of $d(t)$, Shellie's distance at time t. What are the important features of $d(t)$?

Transforming Mathematics Education

mathematics vision project

Tracking the Tortoise

- What conceptual understandings about inverses are solidified in this task?
- What procedural ideas are being developed?

Looking at an entire Functions Module

- Math 1: Modules 3, 4, 5
- Math 2: Modules 1, 2, 3, 4
- Math 3: Module 1, 2, 3, 4, 6

Day 3

Building Procedural Fluency from Conceptual Understanding

"Effective teaching of mathematics builds fluency with procedures on a foundation of conceptual understanding so that students, over time, become skillful in using procedures flexibly as they solve contextual and mathematical problems."

Principles to Actions: Ensuring Mathematical Success for All, National Council Teachers of Mathematics 2014

Building Procedural Fluency from Conceptual Understanding

What are the foundational concepts for the procedures in these tasks?

- Graphing parabolas using transformations
- Completing the square
- Quadratic formula

Building Procedural Fluency from Conceptual Understanding

- How does the conceptual approach based on students' knowledge of functions provide access to the procedures?

- How does the conceptual approach based on students' knowledge of functions provide motivation for the procedures?

What needs to be in place to make this happen in the classroom?

Mathematics Teaching Practices
Establish mathematics goals to focus learning. Effective teaching of mathematics establishes clear goals for the mathematics that students are learning, situates goals within learning progressions, and uses the goals to guide instructional decisions.
Implement tasks that promote reasoning and problem solving. Effective teaching of mathematics engages students in solving and discussing tasks that promote mathematical reasoning and problem solving and allow multiple entry points and varied solution strategies.
Use and connect mathematical representations. Effective teaching of mathematics engages students in making connections among mathematical representations to deepen understanding of mathematics concepts and procedures and as tools for problem solving.
Facilitate meaningful mathematical discourse. Effective teaching of mathematics facilitates discourse among students to build shared understanding of mathematical ideas by analyzing and comparing student approaches and arguments.
Pose purposeful questions. Effective teaching of mathematics uses purposeful questions to assess and advance students' reasoning and sense making about important mathematical ideas and relationships.
Build procedural fluency from conceptual understanding. Effective teaching of mathematics builds fluency with procedures on a foundation of conceptual understanding so that students, over time, become skillful in using procedures flexibly as they solve contextual and mathematical problems.
Support productive struggle in learning mathematics. Effective teaching of mathematics consistently provides students, individually and collectively, with opportunities and supports to engage in productive struggle as they grapple with mathematical ideas and relationships.
Elicit and use evidence of student thinking. Effective teaching of mathematics uses evidence of student thinking to assess progress toward mathematical understanding and to adjust instruction continually in ways that support and extend learning.

Looking at an entire Functions Module

- Math 1: Modules 3, 4, 5
- Math 2: Modules 1, 2, 3, 4
- Math 3: Module 1, 2, 3, 4, 6

Functions Learning Progression: Linear Functions

Linear Functions		
	Secondary Mathematics I	Secondary Mathematics II
Concepts and Definitions		
Constant rate of change	Module 3	
Exchange rate between variables	Module 2	
Arithmetic sequences are discrete linear functions	Module 4	
Comparing linear and quadratic functions		Module 1
Comparing linear and exponential functions	Modules 3 and 4	
Procedures		
Solving systems of linear equations and inequalities	Module 2	
Writing equations of lines given various information	Modules 2, 3, and 4	
Changing the form of a linear equation	Module 4	
Tools		
Graphs of linear functions	Modules 2, 3, and 4	
Story contexts for linear functions	Modules 2, 3, and 4	
Recursive formulas for arithmetic sequences	Module 3	
Point/slope, slope/intercept, and standard form of explicit equations	Module 4	
Tables (including using the first difference)	Modules 2, 3, and 4	

Functions Learning Progression: Exponential Functions

Exponential Functions		
	Secondary Mathematics I	Secondary Mathematics II
Concepts and Definitions		
Constant ratio between terms or growth by equal factors over equal	Module 3 and 4	
intervals	Module 4	
Geometric sequences are discrete exponential functions	Module 4	
Comparing linear and exponential functions		Module 1
Comparing exponential and quadratic functions		
Procedures		
Writing equations of exponential functions	Modules 3 and 4	
Solving basic exponential equations (without using logarithms)	Module 4	
Recognizing equivalent forms and using formulas	Module 4	
Tools		
Graphs of exponential functions	Modules 3 and 4	
Story contexts for exponential functions	Modules 3 and 4	
Recursive formulas for geometric sequences	Module 3	
Explicit equations for exponential functions	Modules 3 and 4	
Tables (including using the first difference)	Modules 3 and 4	

Functions Learning Progression: Quadratic Functions

Quadratic Functions		
	Secondary Mathematics I	Secondary Mathematics II
Concepts and Definitions		
Linear rate of change		Module 1
Product of two linear factors		Module 1
Procedures		
Factoring		Module 2
Completing the Square		Module 2
Graphing using transformations		Module 2
Tools		Modules 1 and 2
Graphs of quadratic functions		Modules 1 and 2
Story contexts for quadratic functions		Module 1
Recursive formulas for quadratic sequences		Module 2
Factored, vertex, and standard form of explicit equations for quadratics		Modules 1 and 2
Tables (including using the first and second differences)		Module 3
Quadratic Formula		

Functions Learning Progression: Inverse Functions and Logarithmic Functions

Inverse Functions and Logarithmic Functions		
	Secondary Mathematics II	Secondary Mathematics III
Concepts and Definitions Functions and their inverses "undo" each other The domain of a function is the range of its inverse Definition of a logarithm	 Module 5	 Module 1 Module 1 Module 1
Procedures Reflecting the graph of a function to find the graph of the inverse Finding inverse functions from the equation of an inverse Using logarithms to solve exponential equations Using properties of logarithms	 Module 5 Module 5	 Module 1 Module 1 Module 2 Module 2
Tools Story contexts for functions and their inverses Graphs of functions and their inverses Tables of functions and their inverses		 Module 1 Module 1 Module 1

Functions Learning Progression: Polynomial and Rational Functions

Polynomial and Rational Functions		
	Secondary Mathematics II	Secondary Mathematics III
Concepts and Definitions		
Rate of change (sum of (n-1) degree polynomial)	Module 1	Module 3
End behavior of polynomial (including quadratic) and rational functions	Module 1	Module 3 and 4
Procedures		
Factoring	Module 2	
Completing the Square (Quadratics)	Module 2	
Finding Roots and Multiplicity		Module 3 and 4
Graphing using transformations	Module 2	
Tools		
Graphs of polynomial and rational functions		Module 3 and 4
Story contexts for polynomial and rational functions		Module 3
Factored form of explicit equations for polynomial and rational functions		Module 3 and 4
Standard form of polynomial functions		Module 3
Tables (including using the first and second differences)		Module 3
Inverse Variation		Module 4

Functions Learning Progression: Other Functions

Other Functions		
	Secondary Mathematics I	Secondary Mathematics II
Concepts and Definitions		
Relationship between variables such that each input has exactly one output	Module 4	Module 4
Domain	Modules 3 and 4	Module 1, 2, 4
Range	Modules 3 and 4	Module 1, 2, 4
Procedures		
Transformation of functions	Module 5	Modules 2 and 4
Tools		
Graphs of functions	Modules 2, 3, 4, 5	Modules 1, 2, 4
Story contexts for functions	Modules 2, 3, 4, 5	Modules 1, 2, 4
Recursive formulas for functions	Module 3	Module 1
Explicit equations using function notation	Module 4, 5	Modules 1, 2, 4
Tables	Modules 2, 3, 4,5	Modules 1, 2, 4

Tracking the Standards

Tracking Student Thinking in MVP Secondary I, Module 5: *Features of Functions*

Cluster titles ⇒	Understand the concept of function and use function notation					Interpret functions that arise in applications in terms of context				Analyze functions using different representations	Build a function that models a relationship between two quantities		Represent and solve equations and inequalities graphically
Description of the nature of student thinking relative to this standard ⇒ in this task ⇓ *(see sample descriptors below*)*	Understand that a function from one set (called the domain) to another set (called the range) assigns to each element of the domain exactly one element of the range (F.IF.1)	If f is a function and x is an element of its domain, then f(x) denotes the output of f corresponding to the input x; the graph of f is the graph of the equation y = f(x) (F.IF.1)	Use function notation and evaluate functions for inputs in their domains (F.IF.2)	Interpret statements that use function notation in terms of a context (F.IF.2)	Recognize that sequences are functions, sometimes defined recursively, whose domain is a subset of the integers (F.IF.3)	For a function that models a relationship between two quantities, interpret key features of graphs and tables in terms of the quantities (F.IF.4)	For a function that models a relationship between two quantities, sketch graphs showing key features given a verbal description of the relationship. (F.IF.4)	Relate the domain of a function to its graph (F.IF.5)	Relate the domain of a function (where applicable) to the quantitative relationship it describes (F.IF.5)	Graph functions expressed symbolically and show key features of the graph (F.IF.7)	Write a function that describes a relationship between two quantities (F.BF.1)	Combine standard function types using arithmetic operations (F.BF.1b)	Explain why the x-coordinates of the points where the graphs of the equations y = f(x) and y = g(x) intersect are the solutions of the equation f(x) = g(x) (A.REI.11)
Getting Ready for a Pool Party	D					D	D	D	D				
Floating Down the River	D					S	S	S	S				
Features of Functions	D				P	P	P	P	P				
The Water Park	D	S	S	S		P	P	P	P	S	D		P

Big ideas about functions from a *Mathematics Vision Project* perspective:

- Families of functions are defined by their rates of change.
- Recursive forms reveal rates of change and cumulative change, are intuitive for students, and are a natural starting point for thinking about functions.
- Each of the representations for functions: table, graph, equation, story context, and geometric figures offer different ways to access, understand, and communicate the ideas of functions.
- A deep understanding of a function includes being able to describe its mathematical features such as x-intercepts, y-intercept, intervals of decrease and increase, end behavior, etc, from any of the representations and to connect the features to the type of function under consideration.

More Big Ideas About Functions

- Functions can all be transformed using the same techniques.
- Procedures and definitions can be formalized based on intuitive understanding of concepts.
- Algebra may be approached from a functions perspective, motivating procedures and anchoring student thinking in the work of functions.
- Understanding of functions grows and builds over time with ideas about more complex functions built upon simpler functions and combinations of functions, both composed or combined using arithmetic operations.
- Functions and their inverses are important models for real-life phenomena.