mathematics vision project

Transforming Mathematics Education

GEOMETRY

A Learning Cycle Approach

Teacher's Notes

MODULE 6

Connecting Algebra and Geometry

MATHEMATICSVISIONPROJECT.ORG

The Mathematics Vision Project

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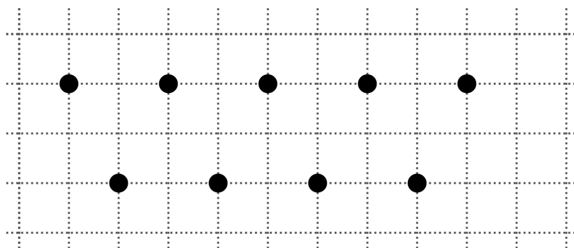
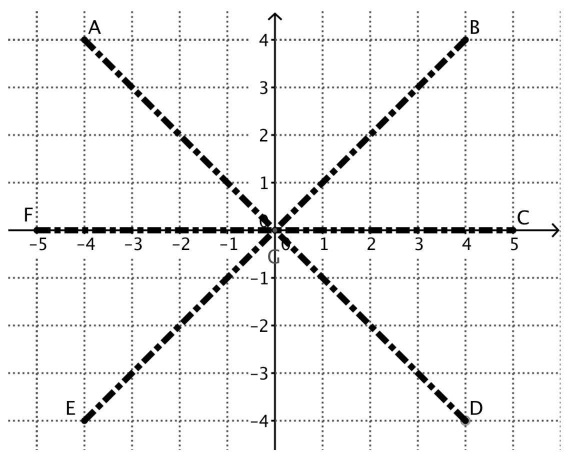
6.1 Go the Distance

A Develop Understanding Task

The performances of the Podunk High School drill team are very popular during half-time at the school's football and basketball games. When the Podunk High School drill team choreographs the dance moves that they will do on the football field, they lay out their positions on a grid like the one below:



In one of their dances, they plan to make patterns holding long, wide ribbons that will span from one dancer in the middle to six other dancers. On the grid, their pattern looks like this:

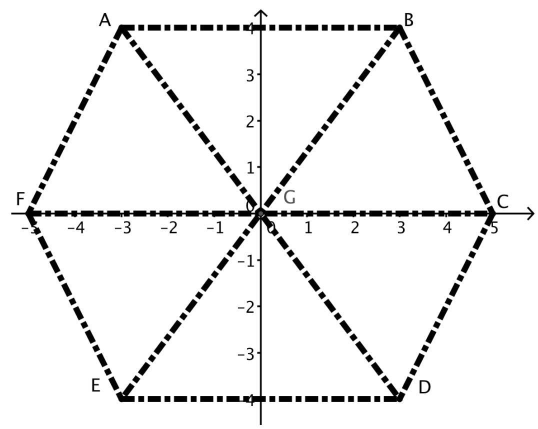


The question the dancers have is how long to make the ribbons. Gabriela (G) is standing in the center and some dancers think that the ribbon from Gabriela (G) to Courtney (C) will be shorter than the one from Gabriela (G) to Brittney (B).

1. How long does each ribbon need to be?

2. Explain how you found the length of each ribbon.

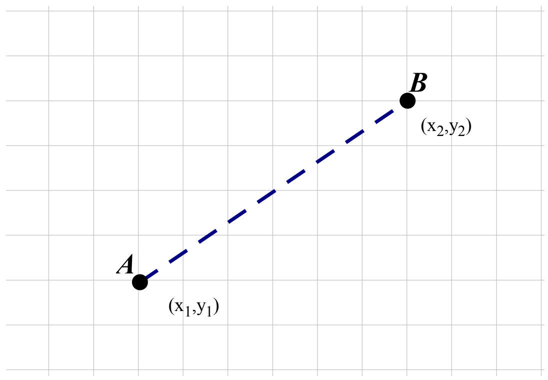
When they have finished with the ribbons in this position, they are considering using them to form a new pattern like this:



3. Will the ribbons they used in the previous pattern be long enough to go between Britney (B) and Courtney (C) in the new pattern? Explain your answer.

Gabriela notices that the calculations she is making for the length of the ribbons reminds her of math class. She says to the group, "Hey, I wonder if there is a process that we could use like what we have been doing to find the distance between any two points on the grid." She decides to think about it like this:

"I'm going to start with two points and draw the line between them that represents the distance that I'm looking for. Since these two points could be anywhere, I named them A (x_1,y_1) and B (x_2,y_2). Hmmmmm. . . . when I figured the length of the ribbons, what did I do next?"



4. Think back on the process you used to find the length of the ribbon and write down your steps here, in terms of (x_1, y_1) and (x_2, y_2).

5. Use the process you came up with in #4 to find the distance between two points located far enough away from each other that using your formula from #4 is more efficient than graphing and counting. For example find the distance between (-11, 25) and (23, -16)

6. Use your process to find the perimeter of the hexagon pattern shown in #3.

6.1 Go the Distance – Teacher Notes

A Develop Understanding Task

Note to Teachers: Calculators facilitate the work for this task.

Purpose: The purpose of this task is to develop the distance formula, based upon students' understanding of the Pythagorean theorem. In the task, students are asked to calculate distances between points using triangles, and then to formalize the process to the distance formula. At the end of the task, students will use the distance formula to find the perimeter of a hexagon.

Core Standards Focus:

G. GPE.4 Use coordinates to prove simple geometric theorems algebraically.

G.GPE.7 Use coordinates to compute perimeters of polygons and areas of triangles and rectangles, e.g., using the distance formula.

Standards for Mathematical Practice:

SMP 1 – Make sense of problems and persevere in solving them.

SMP 7 – Look for and make use of structure.

The Teaching Cycle:

Launch (Whole Class):

Begin the task by ensuring that student understand the problem situation. Project the drawing in #1 and ask students which ribbon looks longer, $\overline{GB}$ or $\overline{GC}$. Ask how they can test their claims. Some students may suggest using the Pythagorean Theorem to find the length of GB. Ask what they would need to use the Pythagorean Theorem. At this point, set students to work on the task.

Explore (Small Group):

During the exploration period, watch for students that are stuck on the first part of the problem. You may ask them to draw the triangle that will help them to use the Pythagorean Theorem and how they might find the length of the legs of the triangle so they can find the hypotenuse. As you monitor student thinking on #3, watch for students who are noticing how to find the length of the legs of the triangle when it has been moved away from the origin. Look for students that have written a good step-by-step procedure for #4. It will probably be difficult for them to use the symbols appropriately, so watch for words that appropriate describe the procedure.

Discuss (Whole Class):

Start the discussion by having a group show how they found the length of $\overline{BC}$ in problem #3. Move next to #4 and have a group that has written a step by step procedure. Try walking through the group's procedure with the numbers from problem #3 and see if it gives the appropriate answer. If necessary, work with the class to modify the procedure so that the list of steps is correct. Once the steps are outlined in words, go through the steps using points A (x_1, y_1) and B (x_2, y_2) and formalize the procedures with the symbols. An example:

Steps in words	**Steps in symbols**
Find the length of the horizontal leg of the triangle	$x_2 - x_1$
Find the length of the vertical leg of the triangle	$y_2 - y_1$
Use the Pythagorean Theorem to write an equation	$(x_2 - x_1)^2 + (y_2 - y_1)^2 = c^2$
Solve for ***c***	$(x_2 - x_1)^2 + (y_2 - y_1)^2 = c^2$
Take the square root of both sides of the equation	$\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} = \sqrt{c^2}$
Simplify	$\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} = c$ (c being the desired distance)

Using algebraic notation to model a correct process that is given verbally will result in deriving the distance formula. After going through this process, apply the formula using the points in #5.

Aligned Ready, Set, Go: Connecting Algebra and Geometry 6.1

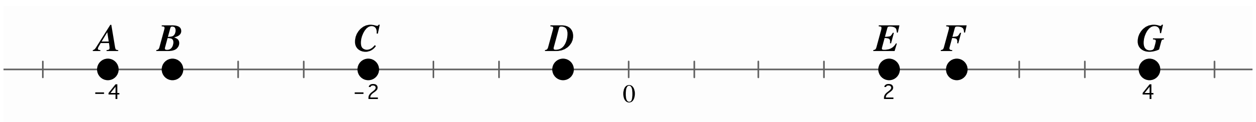
READY, SET, GO! **Name** **Period** **Date**

READY

Topic: Finding the distance between two points

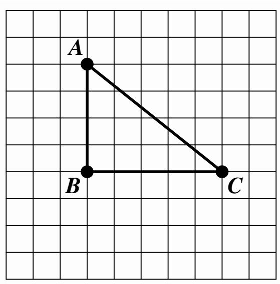
Use the number line to find the distance between the given points. (The notation AB means the distance between the points A and B.)

1. AE 2. CF 3. GB 4. CA 5. BF 6. EG



7. Describe a way to find the distance between two points on a number line without counting the spaces.

8.



a. Find AB.

b. Find BC.

c. Find AC.

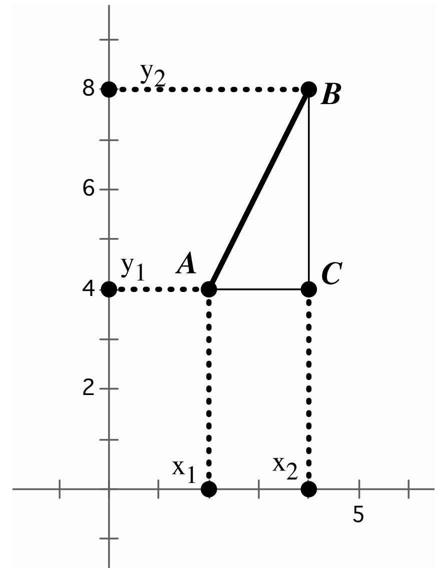
9. Why is it easier to find the distance between point A and point B and point B and point C than it is to find the distance between point A and point C?

10. Explain how to find the distance between point A and point C.

SET

Topic: Slope triangles and the distance formula

Triangle ABC is a slope triangle for the line segment AB where BC is the rise and AC is the run. Notice that the length of segment BC has a corresponding length on the y-axis and the length of AC has a corresponding length on the x-axis. The slope formula is written as $m = \frac{y_2 - y_1}{x_2 - x_1}$ *where m is the slope.*

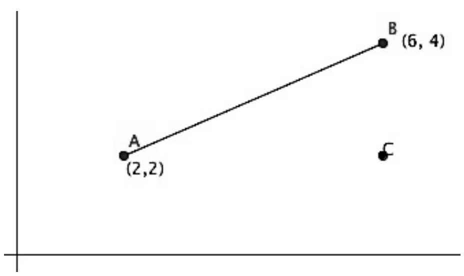


11. a. What does the value $(y_2 - y_1)$ tell you?

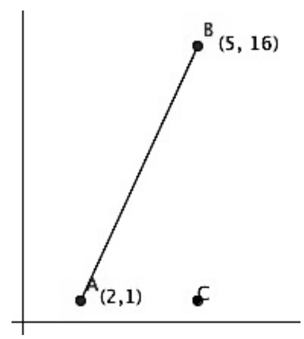
b. What does the value $(x_2 - x_1)$ tell you?

In the previous unit you found the length of a slanted line segment by drawing the slope triangle and then using the Pythagorean theorem on the two sides of the triangle. In this exercise, try to develop a more efficient method of calculating the length of a line segment by using the meaning of $(y_2 - y_1)$ *and* $(x_2 - x_1)$ *combined with the Pythagorean theorem.*

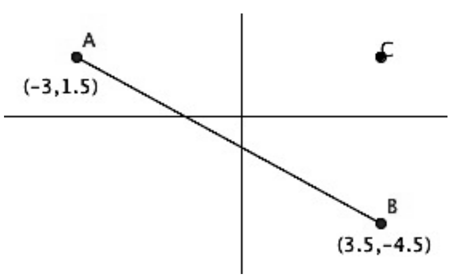
12. Find AB.



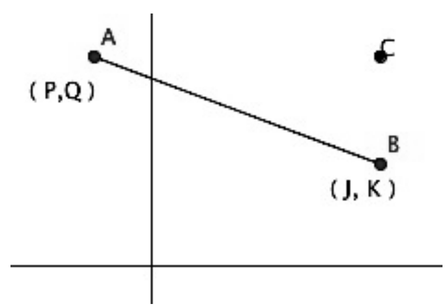
13. Find AB.



14. Find AB.



15. Find AB.



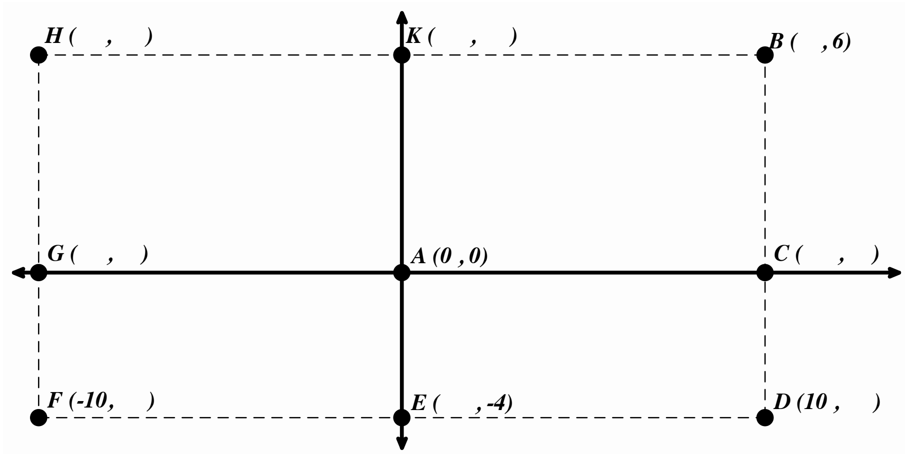
GO

Topic: Rectangular coordinates

Use the given information to fill in the missing coordinates. Then find the length of the indicated line segment.

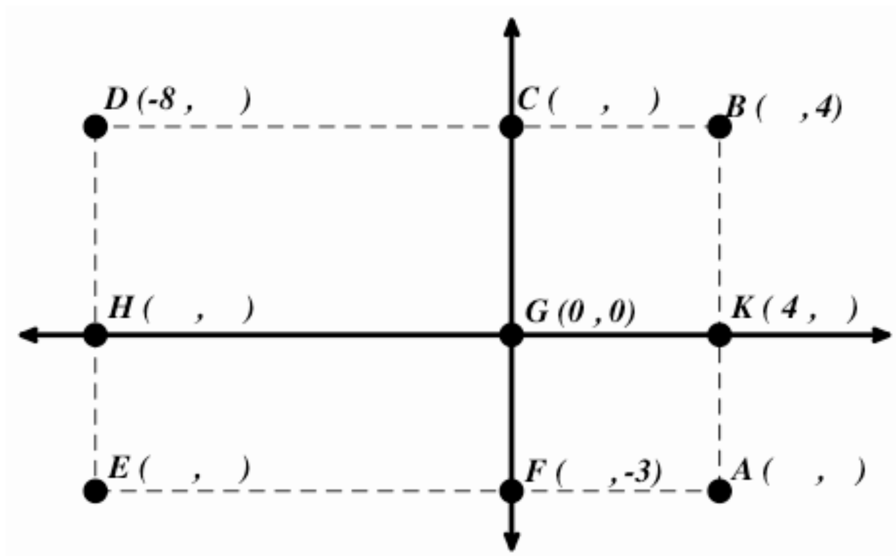
16. a) Find HB.

b) Find BD.



17. a) Find DB

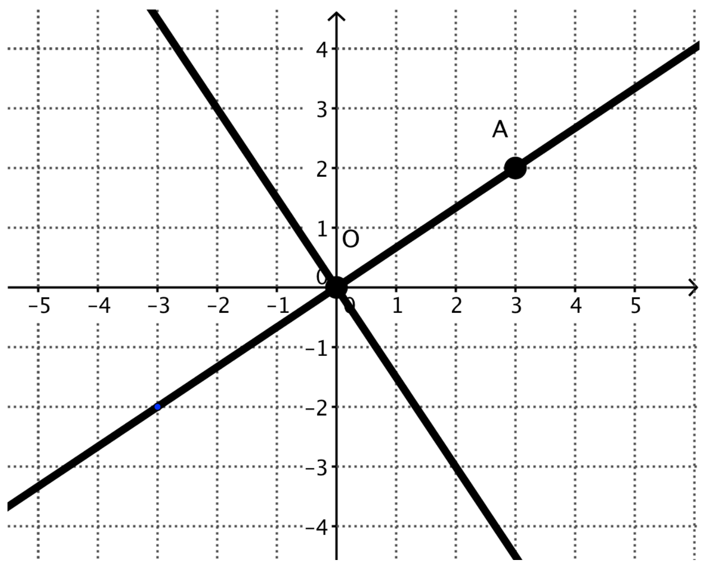
b) Find CF



6.2 Slippery Slopes

A Solidify Understanding Task

While working on "Is It Right?" in a previous module you looked at several examples that lead to the conclusion that the slopes of perpendicular lines are negative reciprocals. Your work here is to formalize this work into a proof. Let's start by thinking about two perpendicular lines that intersect at the origin, like these:



1. Start by drawing a right triangle with the segment $\overline{OA}$ as the hypotenuse. These are often called slope triangles. Based on the slope triangle that you have drawn, what is the slope of $\overleftrightarrow{OA}$?

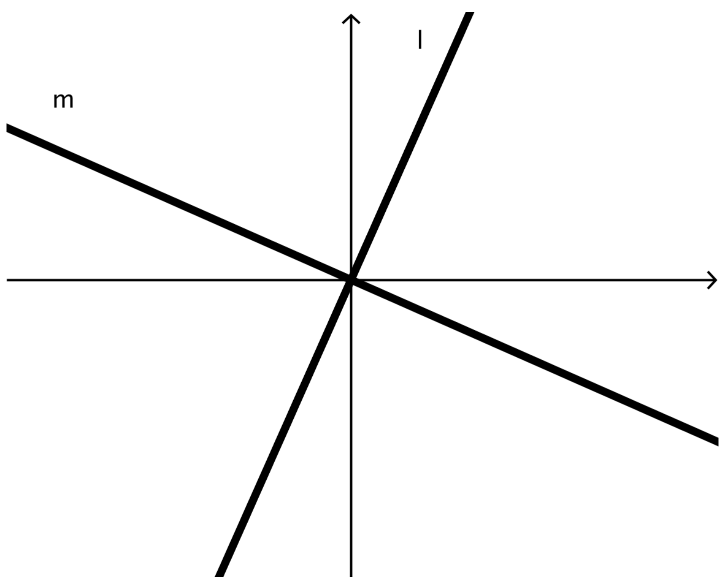
2. Now, rotate the slope triangle 90° about the origin. What are the coordinates of the image of point A?

3. Using this new point, A', draw a slope triangle with hypotenuse $\overline{OA'}$. Based on the slope triangle, what is the slope of the line $\overleftrightarrow{OA'}$?

4. What is the relationship between these two slopes? How do you know?

5. Is the relationship changed if the two lines are translated so that the intersection is at (-5, 7)? How do you know?

To prove a theorem, we need to demonstrate that the property holds for any pair of perpendicular lines, not just a few specific examples. It is often done by drawing a very similar picture to the examples we have tried, but using variables instead of numbers. Using variables represents the idea that it doesn't matter which numbers we use, the relationship stays the same. Let's try that strategy with the theorem about perpendicular lines having slopes that are negative reciprocals.



- Lines *l* and *m* are constructed to be perpendicular.
- Start by labeling a point P on the line *l.*
- Label the coordinates of P.
- Draw the slope triangle from point P.
- Label the lengths of the sides of the slope triangle using variables like *a* and *b* for the run and the rise.

6. What is the slope of line *l*?

Rotate point P 90° about the origin, label it P' and mark it on line m. What are the coordinates of P'?

7. Draw the slope triangle from point P'. What are the lengths of the sides of the slope triangle? How do you know?

8. What is the slope of line *m*?

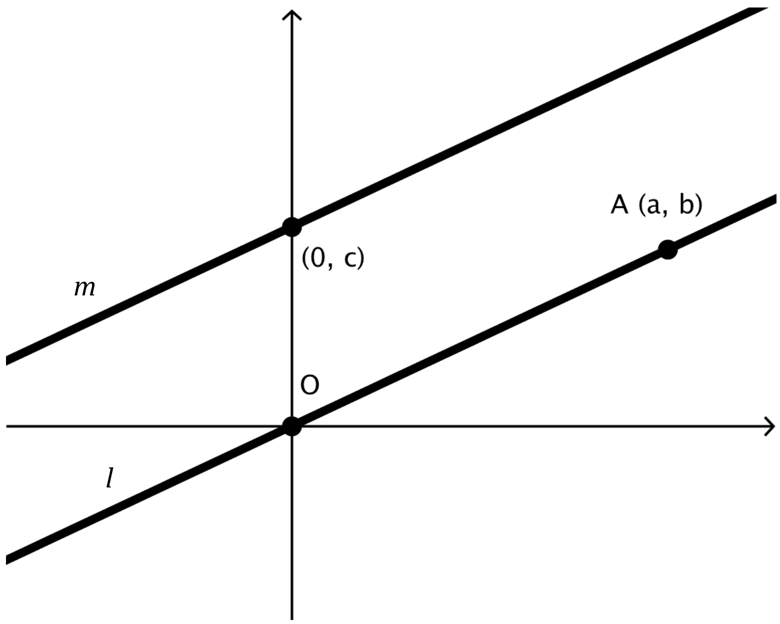
9. What is the relationship between the slopes of line *l* and line *m*? How do you know?

10. Is the relationship between the slopes changed if the intersection between line *l* and line *m* is translated to another location? How do you know?

11. Is the relationship between the slopes changed if lines *l* and *m* are rotated?

12. How do these steps demonstrate that the slopes of perpendicular lines are negative reciprocals for any pair of perpendicular lines?

Think now about parallel lines like the ones below.

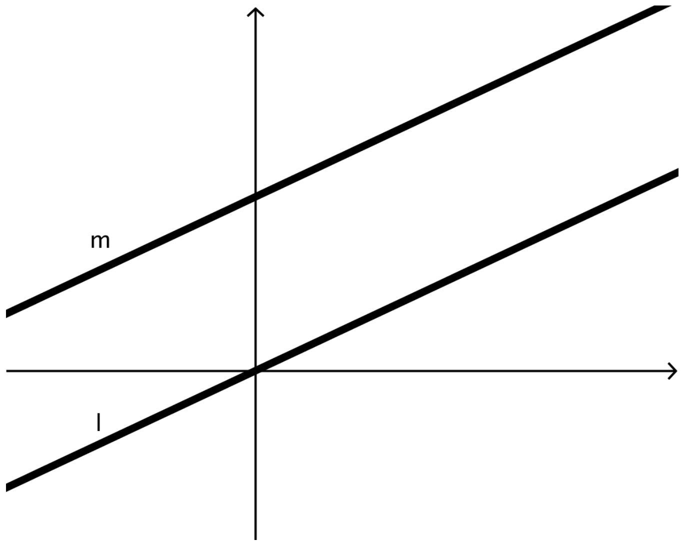


13. Draw the slope triangle from point A to the origin. What is the slope of $\overleftrightarrow{OA}$?

14. What transformation(s) maps the slope triangle with hypotenuse $\overline{OA}$ onto the other line m?

15. What must be true about the slope of line l? Why?

Now you're going to try to use this example to develop a proof, like you did with the perpendicular lines. Here are two lines that have been constructed to be parallel.



16. Show how you know that these two parallel lines have the same slope and explain why this proves that all parallel lines have the same slope.

6.2 Slippery Slopes – Teacher Notes

A Solidify Understanding Task

Purpose: The purpose of this task is to prove that parallel lines have equal slopes and that the slopes of perpendicular lines are negative reciprocals. Students have used these theorems previously. The proofs use the ideas of slope triangles, rotations, and translations. Both proofs are preceded by a specific case that demonstrates the idea before students are asked to follow the logic using variables and thinking more generally.

Core Standards Focus:

G. GPE Use coordinates to prove simple geometric theorems algebraically.

G.GPE.5 Prove the slope criteria for parallel and perpendicular lines and use them to solve geometric problems (e.g., find the equation of a line parallel or perpendicular to a given line that passes through a given point).

Related Standards: G.CO.4, G.CO.5

Standards for Mathematical Practice:

SMP 3 – Construct viable arguments and critique the reasoning of others.

SMP 6 - Attend to precision.

The Teaching Cycle:

Launch (Whole Class):

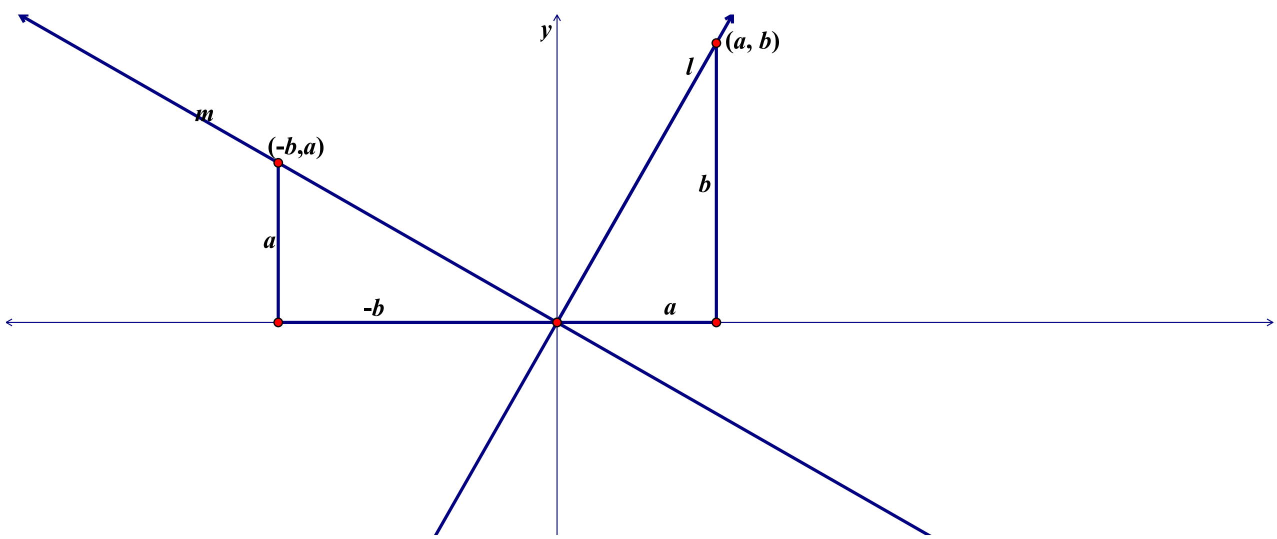
If students haven't been using the term "slope triangle", start the discussion with a brief demonstration of slope triangles and how they show the slope of the line. Students should be familiar with performing a 90 degree rotation from the previous module, so begin the task by having students work individually on questions 1, 2, 3, and 4. When most students have drawn a conclusion for #4, have a discussion of how they know the two lines are perpendicular. Since the purpose is to demonstrate that perpendicular lines have slopes that are negative reciprocals, emphasize that the reason that we know that the lines are perpendicular is that they were constructed based upon a 90 degree rotation.

Explore (Small Group):

The proof that the slopes of perpendicular lines are negative reciprocals follows the same pattern as the example given in the previous problem. Monitor students as they work, allowing them to select a point, label the coordinates and then the sides of the slope triangles. Refer students back to the previous problem, asking them to generalize the steps symbolically if they are stuck. When students are finished with questions 6-12, discuss the proof as a whole group and then have students complete the task.

Discuss (Whole Class):

The setup for the proof is below:



The slope of line l is $\frac{b}{a}$ and the slope of line m is $\frac{a}{-b}$ or - $\frac{a}{b}$. The product of the two slopes is -1, therefore they are negative reciprocals. If the lines are translated so that the intersection is not at the origin, the slope triangles will remain the same. Discuss with the class how questions 6-12 help us to consider all the possible cases, which is necessary in a proof.

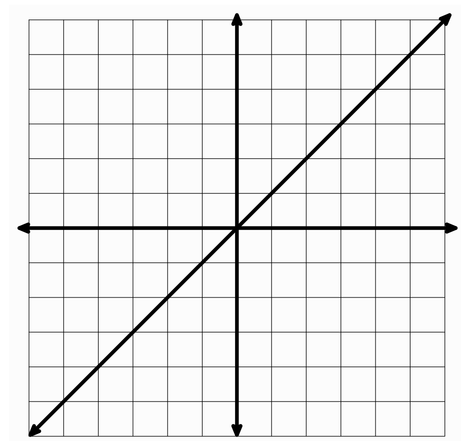
After students have finished the task, go through the brief proof that the slopes of parallel lines are equal.

Aligned Ready, Set, Go: Connecting Algebra and Geometry 6.2

READY, SET, GO! **Name** **Period** **Date**

READY

Topic: Using translations to graph lines



The equation of the line in the graph is $y = x$.

1. a) On the same grid graph a parallel line that is 3 units above it .

   b) Write the equation for the new line in slope-intercept form.

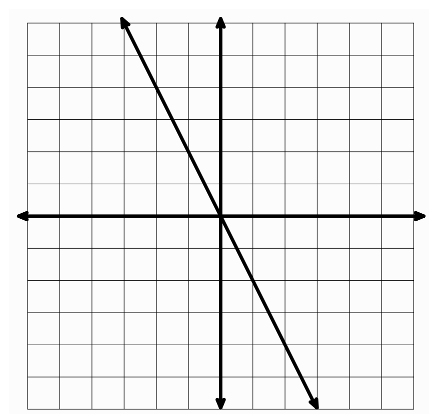
   c) Write the y-intercept of the new line as an ordered pair.

   d) Write the x-intercept of the new line as an ordered pair.

   e) Write the equation of the new line in point-slope form using the *y-intercept.*

   f) Write the equation of the new line in point-slope form using the *x-intercept.*

   g) Explain in what way the equations are the same and in what way they are different.



The graph at the right shows the line $y = -2x$.

2. a) On the same grid, graph a parallel line that is 4 units below it.

   b) Write the equation of the new line in slope-intercept form.

   c) Write the y-intercept of the new line as an ordered pair.

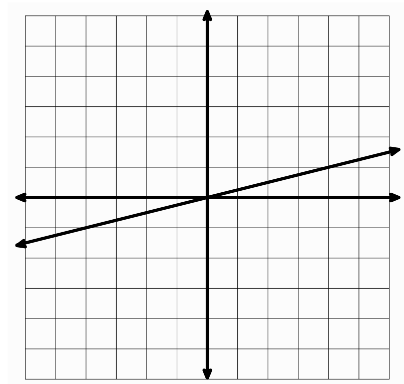
   d) Write the x-intercept of the new line as an ordered pair.

   e) Write the equation of the new line in point-slope form using the *y-intercept.*

   f) Write the equation of the new line in point-slope form using the *x-intercept.*

   g) Explain in what way the equations are the same and in what way they are different.

The graph at the right shows the line $y = \frac{1}{4}x$.



3. a) On the same grid, graph a parallel line that is 2 units below it.

   b) Write the equation of the new line in slope-intercept form.

   c) Write the y-intercept of the new line as an ordered pair.

   d) Write the x-intercept of the new line as an ordered pair.

   e) Write the equation of the new line in point-slope form using the *y-intercept.*

   f) Write the equation of the new line in point-slope form using the *x-intercept.*

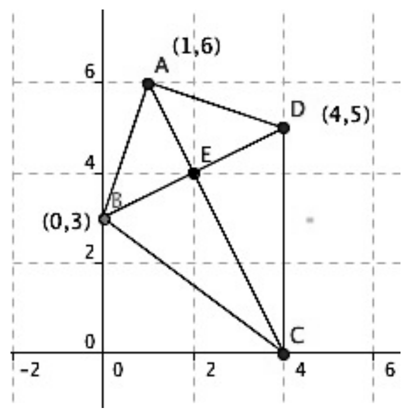
   g) Explain in what way the equations are the same and in what way they are different.

SET

Topic: Verifying and proving geometric relationships

The quadrilateral at the right is called a **kite.**
Complete the mathematical statements about the kite using the given symbols. Prove each statement algebraically. (A symbol may be used more than once.)

$\cong \quad \perp \quad \parallel \quad < \quad > \quad =$



Proof

4. $\overline{BC}$_________$\overline{DC}$ ____

5. $\overline{BD}$_________$\overline{AC}$ ____

6. $\overline{AB}$_________$\overline{BC}$ ____

7. ΔABC_____ ΔADC ____

____

8. $\overline{BE}$________$\overline{ED}$ ____

____

9. $\overline{AE}$________$\overline{ED}$ ____

____

10. $\overline{AC}$________$\overline{BD}$ ____

____

GO

Topic: Writing equations of lines

Use the given information to write the equation of the line in standard form. $(\boldsymbol{Ax + By = C})$

11. $\boldsymbol{Slope: -\frac{1}{4} \quad point\ (12, 5)}$

12. $\boldsymbol{P(11, -3),\ Q(6, 2)}$

13. $\boldsymbol{x - intercept: -2;\ \ y - intercept: -3}$

14. $\boldsymbol{All\ x\ values\ are\ (-7).\ Y\ is\ any\ number.}$

15. $\boldsymbol{Slope: \frac{1}{2};\ \ x - intercept: 5}$

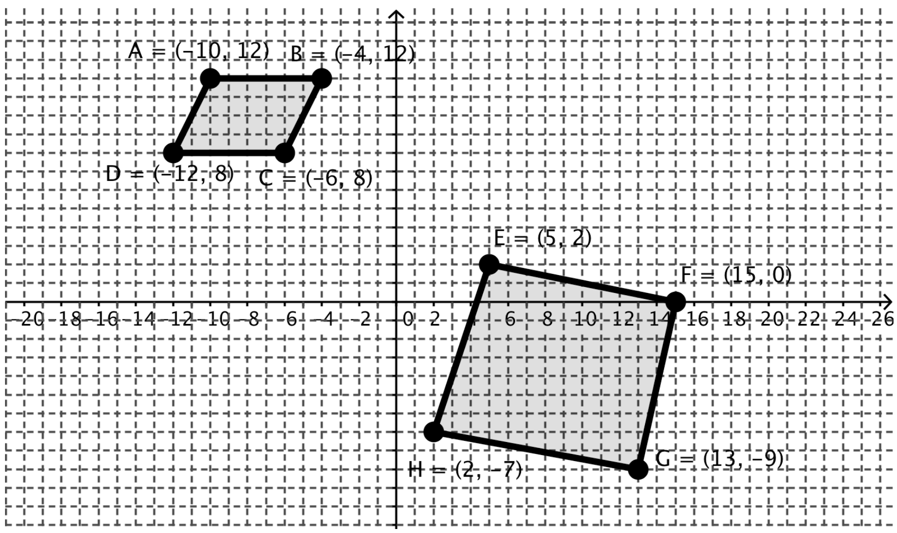
16. $\boldsymbol{E\ (-10, 17),\ G\ (13, 17)}$

6.3 Prove It!

A Practice Understanding Task

In this task you need to use all the things you know about quadrilaterals, distance, and slope to prove that the shapes are parallelograms, rectangles, rhombi, or squares. Be systematic and be sure that you give all the evidence necessary to verify your claim.

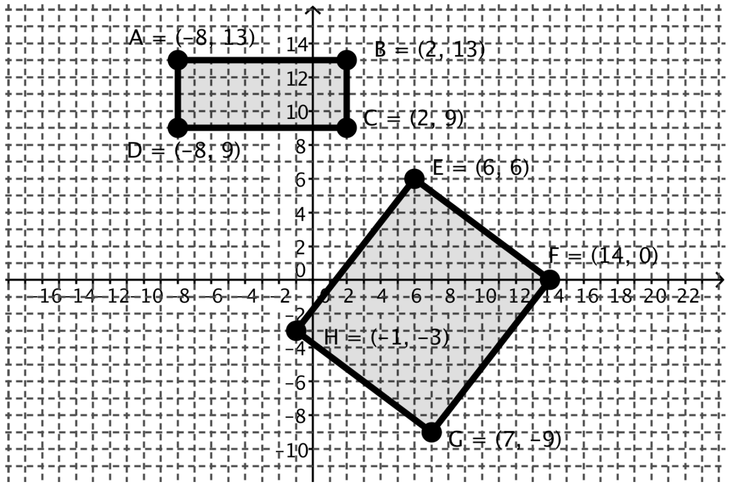
1.



a. Is ABCD a parallelogram? Explain how you know.

b. Is EFGH a parallelogram? Explain how you know.

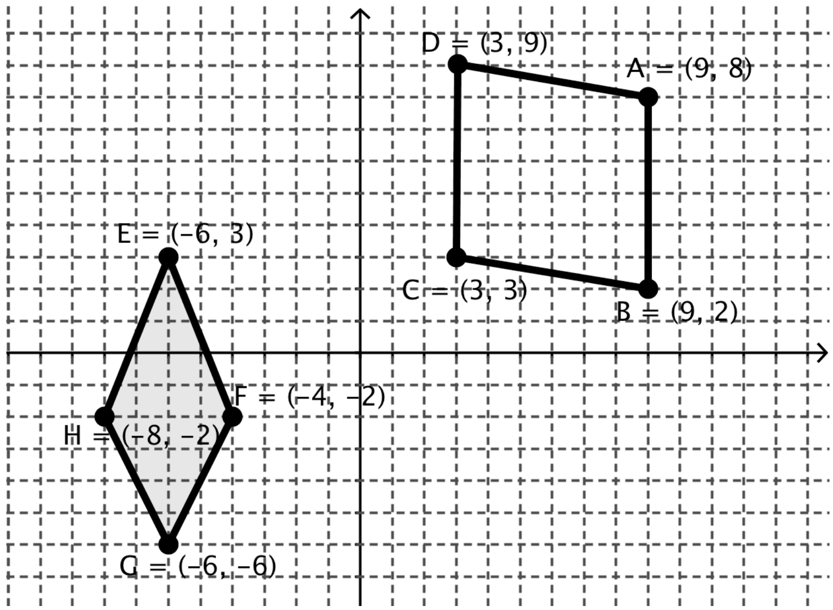
2.



a. Is ABCD a rectangle? Explain how you know.

b. Is EFGH a rectangle? Explain how you know.

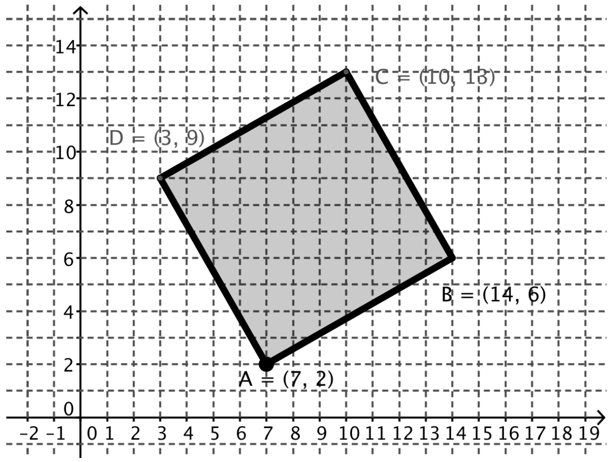
3.



a. Is ABCD a rhombus? Explain how you know.

b. Is EFGH a rhombus? Explain how you know.

4.



a. Is ABCD a square? Explain how you know.

6.3 Prove It! – Teacher Notes

A Practice Understanding Task

Purpose: The purpose of this task is to solidify student understanding of quadrilaterals and to connect their understanding of geometry and algebra. In the task they will use slopes and distance to show that particular quadrilaterals are parallelograms, rectangles, rhombi or squares. This task will also strengthen student understanding of justification and proof, and the need to put forth a complete argument based upon sound mathematical reasoning.

Core Standards Focus:
G.GPE.4 Use coordinates to prove simple geometric theorems algebraically. For example, prove or disprove that a figure defined by four given points in the coordinate plane is a rectangle.

Related Standards: G.GPE.5, G.GPE.7

Standards for Mathematical Practice:

SMP 3 – Construct viable arguments and critique the reasoning of others
SMP 6 – Attend to precision

The Teaching Cycle:
Launch (Whole Class):
Launch the task with a discussion of what students know about the properties of quadrilaterals, for instance that a rhombus has two pairs of parallel sides (making it a parallelogram), congruent sides, and perpendicular diagonals. Discuss what you would need to show to prove a claim that a figure is a particular quadrilateral. For instance, it is not enough to show that a shape is a rhombus by showing that the two pairs of sides are parallel, but it would be enough to show that the diagonals are perpendicular. Why?
Explore (Small Group):
Monitor students as they work. It may be helpful to recognize that each set of problems is set up so there is a simple case and a more complicated case. The simple case is designed to help students

get ideas for how to prove the more complicated case. Keep track of the various approaches that students use to verify their claims and press them to organize their work so that it communicates to an outside observer. Students should be showing sides or diagonals are parallel or perpendicular using the slope properties, and the distance formula to show that sides or diagonals are congruent. You may also see students try to show one figure to be a particular quadrilateral and then use transformations to show that the second figure is the same type. Select one group of students that have articulated a clear argument for each type of quadrilateral. Be sure to also select a variety of approaches so that students have opportunity to make connections and become more fluent.

Discuss (Whole Class):

The discussion should proceed in the same order as the task, with different groups demonstrating their strategies for one parallelogram, one rectangle, one rhombus, and one square. Select the shape that does not have sides that are on a grid line, so that students are demonstrating the more challenging cases. One recommended sequence for the discussion would be:

1b. Quadrilateral EFGH is not a parallelogram demonstrated by showing that one pair of opposite sides are not parallel using slopes.

2b. Rectangle EFGH demonstrated by using the distance formula to show that the diagonals are congruent.

3a. Showing quadrilateral ABCD is a rhombus because the diagonals are perpendicular and the sides are congruent (or that the sides are congruent and the opposite sides are parallel).

3b. Showing that quadrilateral EFGH is not a rhombus because the sides are not congruent . Ask if the figure is a parallelogram? How do we know?

4a. Showing that quadrilateral ABCD is a square because adjacent sides are perpendicular and sides are congruent. Ask if it sufficient to show that the sides are congruent? How do we know that the opposite sides are parallel?

Aligned Ready, Set, Go: Connecting Algebra and Geometry 6.3

READY, SET, GO! **Name** **Period** **Date**

READY

Topic: Interpreting tables of value as ordered pairs.

Find the value of $f(x)$ for the given domain. Write x and $f(x)$ as an ordered pair.

1. $f(x) = 3x - 2$

x	$f(x)$	$(x, f(x))$
-2		
-1		
0		
1		
2		

2. $f(x) = x^2$

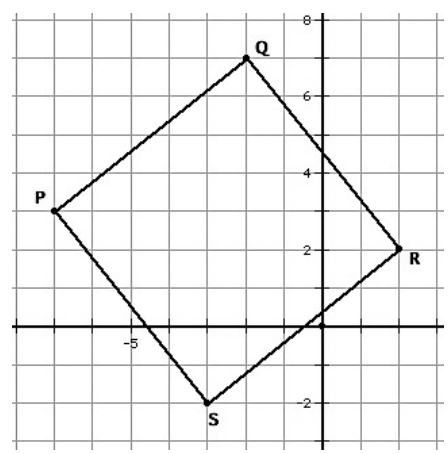
x	$f(x)$	$(x, f(x))$
-2		
-1		
0		
1		
2		

3. $f(x) = 5^x$

x	$f(x)$	$(x, f(x))$
-2		
-1		
0		
1		
2		

SET

Topic: Identifying specific quadrilaterals



4. a) Is the figure at the right a rectangle? Justify your answer.

b) Is the figure at the right a rhombus? Justify your answer.

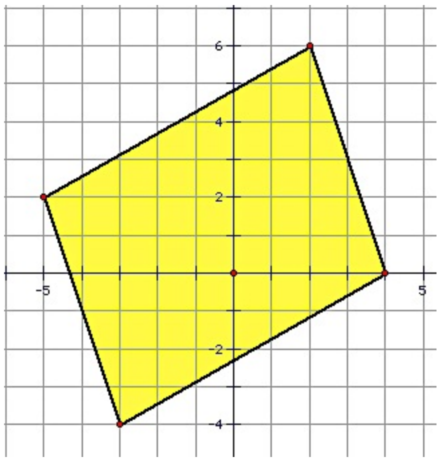
c) Is the figure at the right a square? Justify your answer.

GO

Topic: Calculating perimeters of geometric shapes

Find the perimeter of each figure below. Round answers to the nearest hundredth.

5.



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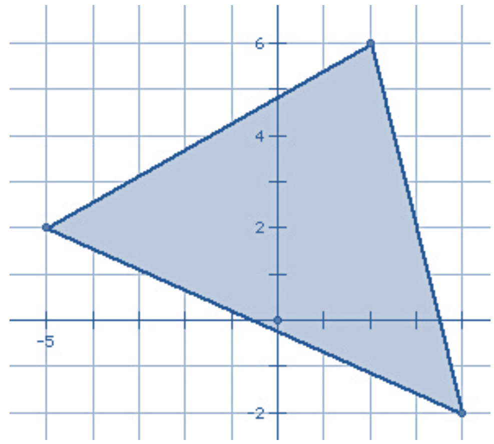

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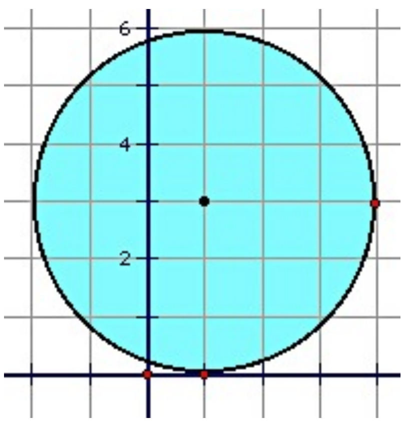

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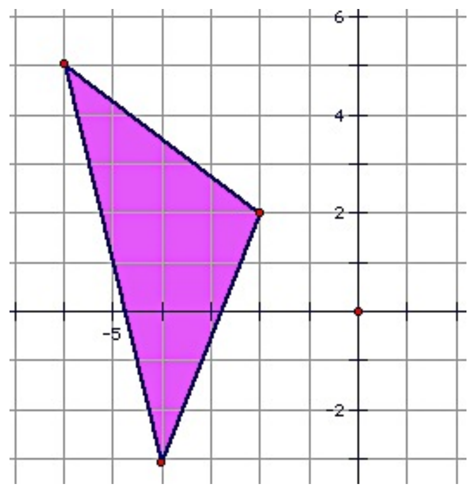

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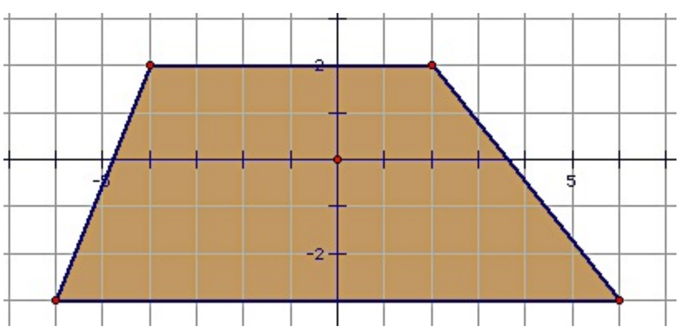

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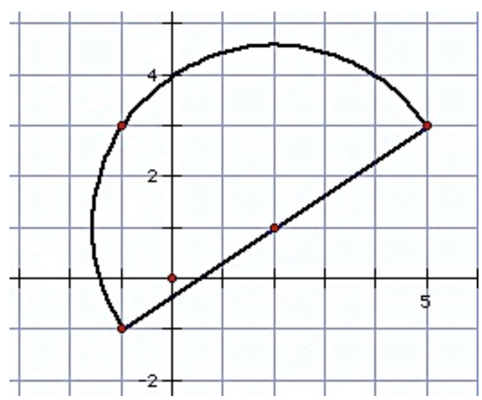
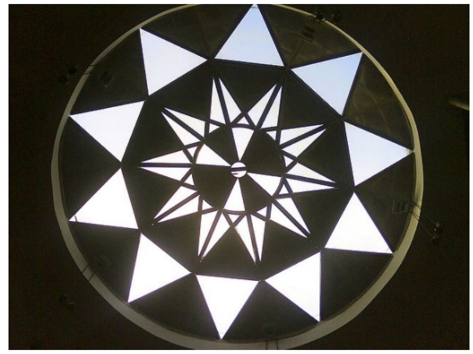


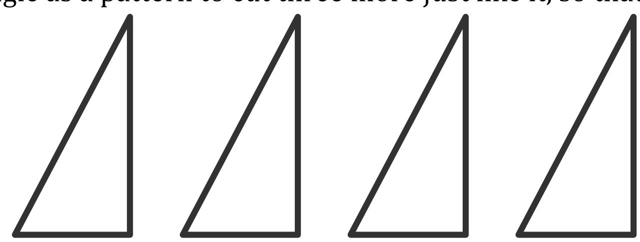
6.4 Circling Triangles (Or Triangulating Circles)



A Develop Understanding Task

Using the corner of a piece of colored paper and a ruler, cut a right triangle with a 6” hypotenuse, like so:

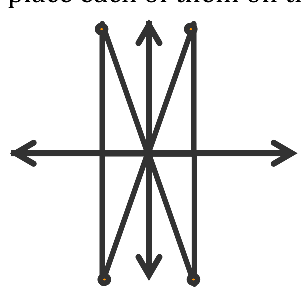
Use this triangle as a pattern to cut three more just like it, so that you have a total of four congruent triangles.



1. Choose one of the legs of the first triangle and label it *x* and label the other leg *y*. What is the relationship between the three sides of the triangle?

2. When you are told to do so, take your triangles up to the board and place each of them on the coordinate axis like this:

   Mark the point at the end of each hypotenuse with a pin.



3. What shape is formed by the pins after the class has posted all of their triangles? Why would this construction create this shape?

4. What are the coordinates of the pin that you placed in:
   a. the first quadrant?
   b. the second quadrant?
   c. the third quadrant?
   d. the fourth quadrant?

5. Now that the triangles have been placed on the coordinate plane, some of your triangles have sides that are of length $-x$ or $-y$. Is the relationship $x^2 + y^2 = 6^2$ still true for these triangles? Why or why not?

6. What would be the equation of the graph that is the set on all points that are 6" away from the origin?

7. Is the point (0, -6) on the graph? How about the point (3, 5.193)? How can you tell?

8. If the graph is translated 3 units to the right and 2 units up, what would be the equation of the new graph? Explain how you found the equation.

6.4 Circling Triangles (Or Triangulating Circles) – Teacher Notes

A Develop Understanding Task

Purpose: This purpose of this task is for students to connect their geometric understanding of circles as the set of all point equidistant from a center to the equation of a circle. In the task, students construct a circle using right triangles with a radius of 6 inches. This construction is intended to focus students on the Pythagorean Theorem and to use it to generate the equation of a circle centered at the origin. After constructing a circle at the origin, students are asked to use their knowledge of translations to consider how the equation would change if the center of the circle is translated.

Core Standards Focus:
G-GPE Expressing Geometric Properties with Equations
Translate between the geometric description and the equation for a conic section
G-GPE.1 Derive the equation of a circle of given center and radius using the Pythagorean Theorem; complete the square to find the center and radius of a circle given by an equation.

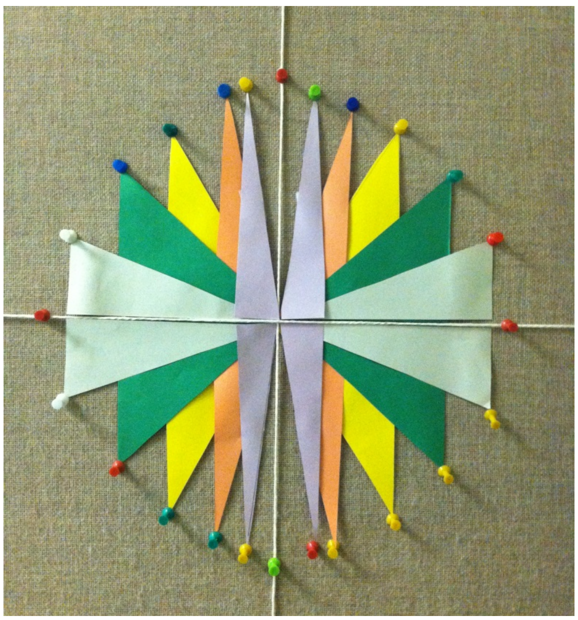
Standards for Mathematical Practice:

SMP 1 – Make sense of problems and persevere in solving them
SMP 7 – Look for and make use of structure

The Teaching Cycle:

Launch (Whole Class):
Be prepared for the class activity by having at least two sheets of colored paper (heavy paper is better), rulers and scissors for students to use. On a board in the classroom, create the coordinate axes with strings or tape. Prepare a way for students to mark the endpoint of their triangles with a

tack or some other visible mark so that the circle that will be constructed is visible. Depending on the size of your class, you may choose to have several axes set up and divide students into groups. Ask students to follow the instructions on the first page and post their triangles. Encourage some students to select the longest leg of the triangle to be x and others to select the shortest leg to be x so that there are as many different points on the circle formed as possible. Watch as students post their triangles to see that they get all four of them into the proper positions. An example of what the board will look like when the triangles are posted is:



Tell students to work on problem #3 as other students finish posting their triangles. When all students are finished, ask students why the shape formed is a circle. They should be able to relate the idea that since each triangle had a hypotenuse of 6", they formed a circle that has a radius of 6". Use a 6" string to demonstrate how the radius sweeps around the circle, touching the endpoint of each hypotenuse.

Explore (Small Group):

Ask students to work on the remaining questions. Monitor student work to support their thinking about the Pythagorean Theorem using x and y as the lengths of the legs of any of the right triangles used to form the circle. Question #5 may bring about confusion about the difference between $(-x)^2$

and $-x^2$. Remind students that in this case, x is a positive number, so $-x$ is a negative number, and the square of a negative number is positive.

Discuss (Whole Class):

Begin the discussion with #6. Ask students for their equation and how their equation represents all the points on the circle. Press for students to explain how the equation works for points that lie in quadrants II, III, and IV.

Turn the discussion to #7. Ask how they decided if the points were on the circle. Some students may have tried measuring or estimating, so be sure that the use of the equation is demonstrated. After discussing the point (3, 5.193), ask students what could be said about (3, -5.193) or (-3, -5.193) to highlight the symmetries and how they come up in the equation.

Finally, discuss the last question. Students should have various explanations for the change in the equation. Some may use the patterns they have observed in shifting functions, although it should be noted that this graph is not a function. Other students may be able to articulate the idea that $x - 3$ represents the length of the horizontal side of the triangle that was originally length x, now that it has been moved three units to the right.

Aligned Ready, Set, Go: Connecting Algebra and Geometry 6.4

READY, SET, GO! Name Period Date

READY

Topic: Factoring special products

Factor the following as the difference of 2 squares or as a perfect square trinomial. Do not factor if they are neither.

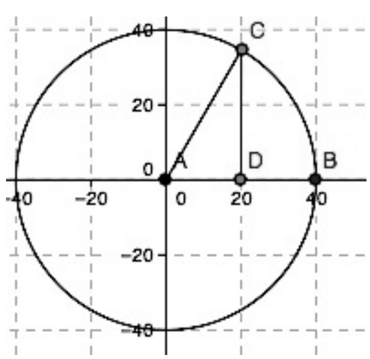
$b^2 - 49$	$b^2 - 2b + 1$	$b^2 + 10b + 25$
$x^2 - y^2$	$x^2 - 2xy + y^2$	$25x^2 - 49y^2$
$36x^2 + 60xy + 25y^2$	$81a^2 - 16d^2$	$144x^2 - 312xy + 169y^2$

SET

Topic: Writing the equations of circles

Write the equation of each circle centered at the origin.

10.



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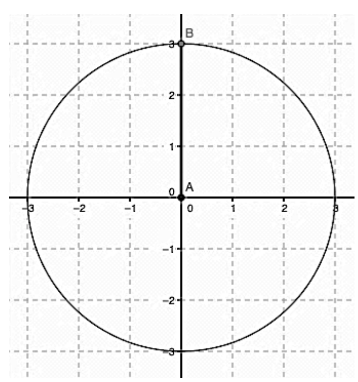

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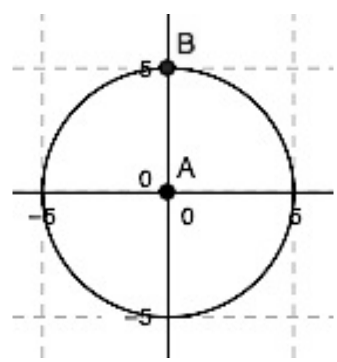

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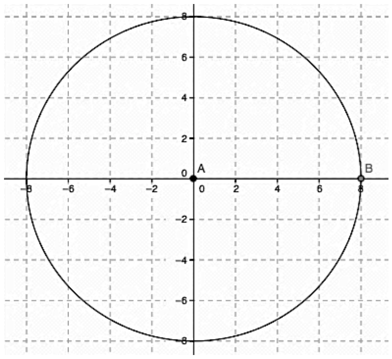

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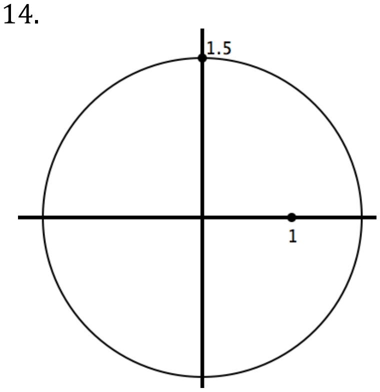

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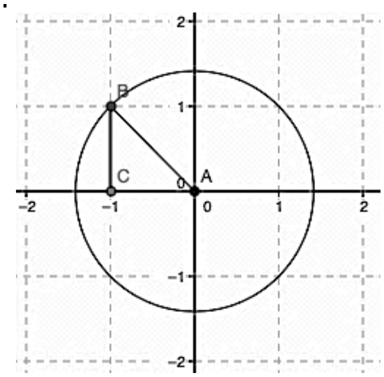

GO

Topic: Verifying Pythagorean triples

Identify which sets of numbers could be the sides of a right triangle. Show your work.

16. $\{9, 12, 15\}$

17. $\{9, 10, \sqrt{19}\}$

18. $\{1, \sqrt{3}, 2\}$

19. $\{2, 4, 6\}$

20. $\{\sqrt{3}, 4, 5\}$

21. $\{10, 24, 26\}$

22. $\{\sqrt{2}, \sqrt{7}, 3\}$

23. $\{2\sqrt{2}, 5\sqrt{3}, 9\}$

24. $\{4ab^3\sqrt{10}, 6ab^3, 14ab^3\}$

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6.5 Getting Centered

A Solidify Understanding Task

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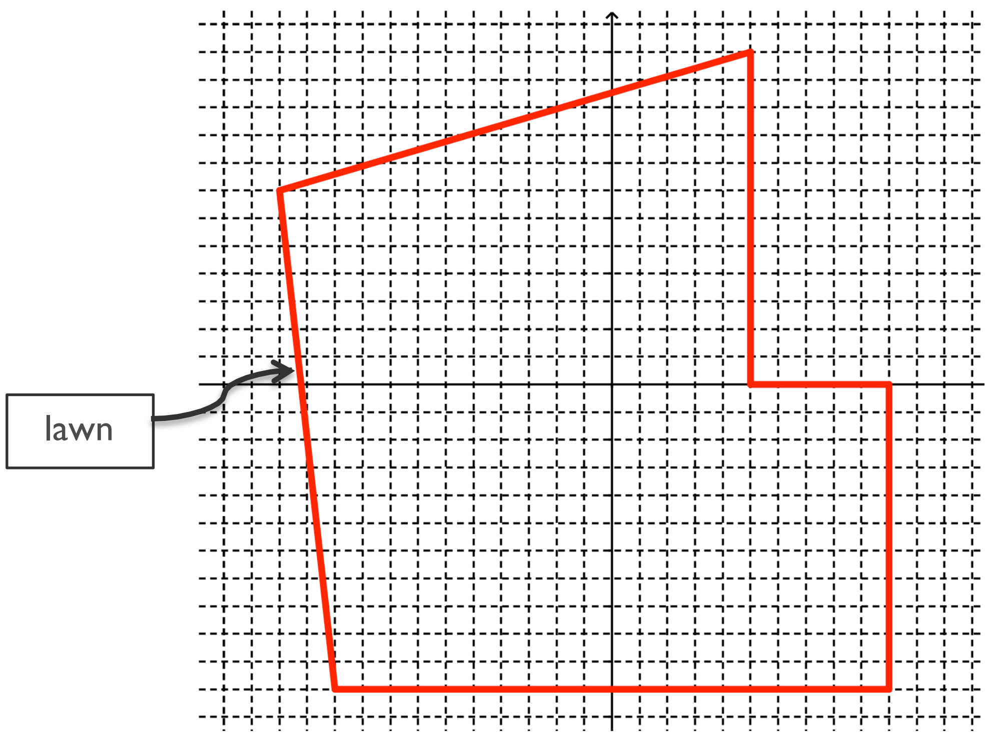
Malik's family has decided to put in a new sprinkling system in their yard. Malik has volunteered to lay the system out. Sprinklers are available at the hardware store in the following sizes:

- Full circle, maximum 15' radius
- Half circle, maximum 15' radius
- Quarter circle, maximum 15' radius

All of the sprinklers can be adjusted so that they spray a smaller radius. Malik needs to be sure that the entire yard gets watered, which he knows will require that some of the circular water patterns will overlap. He gets out a piece of graph paper and begins with a scale diagram of the yard. In this diagram, the length of the side of each square represents 5 feet.



1. As he begins to think about locating sprinklers on the lawn, his parents tell him to try to cover the whole lawn with the fewest number of sprinklers possible so that they can save some money. The equation of the first circle that Malik draws to represent the area watered by the sprinkler is:

   $$(x+25)^2+(y+20)^2=225$$

   Draw this circle on the diagram using a compass.

2. Lay out a possible configuration for the sprinkling system that includes the first sprinkler pattern that you drew in #1.

3. Find the equation of each of the full circles that you have drawn.

Malik wrote the equation of one of the circles and just because he likes messing with the algebra, he did this:

Original equation: $(x-3)^2+(y+2)^2 \ = 225$

$$x^2 - 6x + 9 + \ y^2 + 4y + 4 = 225$$

$$x^2 + \ y^2 - 6x + 4y - 212 = 0$$

Malik thought, "That's pretty cool. It's like a different form of the equation. I guess that there could be different forms of the equation of a circle like there are different forms of the equation of a parabola or the equation of a line." He showed his equation to his sister, Sapana, and she thought he was nuts. Sapana, said, "That's a crazy equation. I can't even tell where the center is or the length of the radius anymore." Malik said, "Now it's like a puzzle for you. I'll give you an equation in the new form. I'll bet you can't figure out where the center is."

Sapana said, "Of course, I can. I'll just do the same thing you did, but work backwards."

4. Malik gave Sapana this equation of a circle:

$$x^2 + y^2 - 4x + 10y + 20 = 0$$

Help Sapana find the center and the length of the radius of the circle.

5. Sapana said, "Ok. I made one for you. What's the center and length of the radius for this circle?"

$$x^2 + y^2 + 6x - 14y - 42 = 0$$

6. Sapana said, "I still don't know why this form of the equation might be useful. When we had different forms for other equations like lines and parabolas, each of the various forms highlighted different features of the relationship." Why might this form of the equation of a circle be useful?

$$x^2 + y^2 + Ax + By + C = 0$$

6.5 Getting Centered – Teacher Notes

A Solidify Understanding Task

Purpose: The purpose of this task is to solidify understanding of the equation of the circle. The task begins with sketching circles and writing their equations. It proceeds with the idea of squaring the $(x - h)^2$ and $(y - k)^2$ expressions to obtain a new form of an equation. Students are then challenged to reverse the process to find the center of the circle.

Core Standards Focus:

G-GPE Expressing Geometric Properties with Equations

Translate between the geometric description and the equation for a conic section

G-GPE.1 Derive the equation of a circle of given center and radius using the Pythagorean Theorem; complete the square to find the center and radius of a circle given by an equation.

Standards for Mathematical Practice:

SMP 2 – Reason abstractly and quantitatively

SMP 8 – Look for express regularity in repeated reasoning

The Teaching Cycle:

Launch (Whole Class):

Begin the task by helping students to understand the context of developing a diagram for a sprinkling system. The task begins with students drawing circles to cover the yard and writing equations for the circles that they have sketched. Allow students some time to work to make their diagrams and write their equations. However, don't spend too much time trying to completely cover the lawn. The point is to draw four or more circles and to write their equations. As students are working, be sure that they are accounting for the scale as they name the center of their circles. Ask several students to share their equations. After each student shares, ask the class to identify the center and radius of the equation. After several students have shared, ask one student to take the last equation shared and square the $(x - h)^2$ and $(y - k)^2$ expressions and simplify the remaining equation. Tell students that this is what Malik did and now their job is to take the equation back to the form in which they can easily read the center and radius.

Explore (Small Group):
Since students have previously completed the square for parabolas, some students will think to apply the same process here. Monitor their work, watching for groups that have different answers for the same equation (hopefully, one of them is correct).

Discuss (Whole Class):
Begin the discussion by posting two different equations that answer question #4. Ask students how they can decide which equation is correct. They may suggest working backwards to the original equation, or possibly checking a point. Decide which equation is correct and ask that group to describe the process they used to get the answer. Ask another group that has a correct version of #5 to show how they obtained their answer. You may also wish to discuss #6. Wrap up the lesson up by working with the class to create a set of steps that they can follow to get the equation back to center/radius form.

Aligned Ready, Set, Go: Connecting Algebra and Geometry 6.5

READY, SET, GO! Name Period Date

READY

Topic: Making perfect square trinomials

Fill in the number that completes the square. Then write the trinomial in factored form.

1. x^2+6x+ __________
2. x^2-14x+ __________
3. x^2-50x+ __________
4. x^2-28x+ __________

On the next set, leave the number that completes the square as a fraction. Then write the trinomial in factored form.

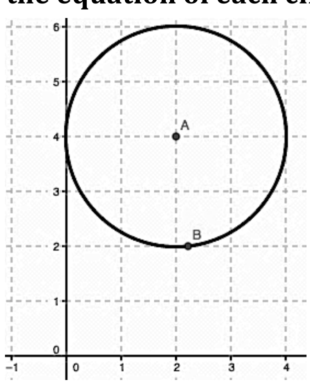
5. x^2-11x+ __________
6. x^2+7x+ __________
7. x^2+15x+ __________
8. $x^2+\frac{2}{3}x+$ __________
9. $x^2-\frac{1}{5}x+$ __________
10. $x^2-\frac{3}{4}x+$ __________

SET

Topic: Writing equations of circles with center (h, k) and radius *r*.

Write the equation of each circle.

11.



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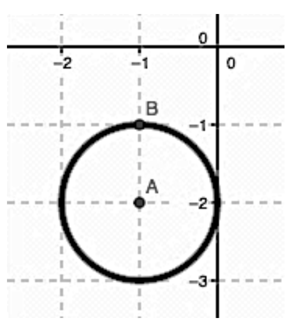

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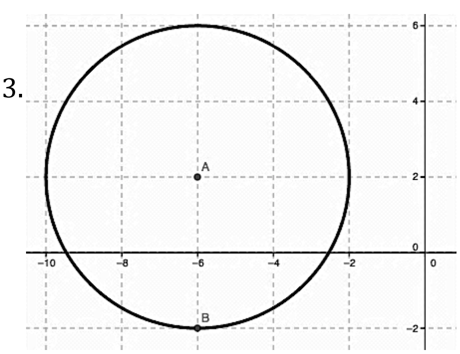


Write the equation of the circle with the given center and radius. Then write it in expanded form.

14. Center: (5, 2) Radius: 13

15. Center: (-6, -10) Radius: 9

16. Center: (0, 8) Radius: 15

17. Center: (19, -13) Radius: 1

18. Center: (-1, 2) Radius: 10

19. Center: (-3, -4) Radius: 8

Go

Topic: Verifying if a point is a solution

Identify which point is a solution to the given equation. Show your work.

20. $y = \frac{4}{5}x - 2$ a. (-15, -14) b. (10, 10)

21. $y = 3|x|$ a. (-4, -12) b. $(-\sqrt{5}, 3\sqrt{5})$

22. $y = x^2 + 8$ a. $(\sqrt{7}, 15)$ b. $(\sqrt{7}, -1)$

23. $y = -4x^2 + 120$ a. $(5\sqrt{3}, -180)$ b. $(5\sqrt{3}, 40)$

24. $x^2 + y^2 = 9$ a. $(8, -1)$ b. $(-2, \sqrt{5})$

25. $4x^2 - y^2 = 16$ a. $(-3, \sqrt{10})$ b. $(-2\sqrt{2}, 4)$

6.6 Circle Challenges

A Practice Understanding Task

Once Malik and Sapana started challenging each other with circle equations, they got a little more creative with their ideas. See if you can work out the challenges that they gave each other to solve. Be sure to justify all of your answers.

1. Malik's challenge:

   What is the equation of the circle with center (-13,-16) and containing the point (-10,-16) on the circle?

2. Sapana's challenge:

   The points (0, 5) and (0,-5) are the endpoints of the diameter of a circle. The point (3, y) is on the circle. What is a value for y?

3. Malik's challenge:

   Find the equation of a circle with center in the first quadrant and is tangent to the lines $x = 8$, $y = 3$, and $x = 14$.

4. Sapana's challenge:

   The points (4,-1) and (-6,7) are the endpoints of the diameter of a circle. What is the equation of the circle?

5. Malik's challenge:

   Is the point (5,1) inside, outside, or on the circle $x^2 - 6x + y^2 + 8y = 24$? How do you know?

6. Sapana's challenge:

   The circle defined by $(x-1)^2 + (y+4)^2 = 16$ is translated 5 units to the left and 2 units down. Write the equation of the resulting circle.

7. Malik's challenge:

   There are two circles, the first with center (3,3) and radius r_1, and the second with center (3, 1) and radius r_2.

   a. Find values r_1 and r_2 of so that the first circle is completely enclosed by the second circle.

   b. Find one value of r_1 and one value of r_2 so that the two circles intersect at two points.

   c. Find one value of r_1 and one value of r_2 so that the two circles intersect at exactly one point.

6.6 Circle Challenges – Teacher Notes

A Practice Understanding Task

Purpose:
The purpose of this task is for students to practice using the equation of the circle in different ways. In each case, they must draw inferences from the information given and use the information to find the equation of the circle or to justify conclusions about the circle. They will use the distance formula to find the measure of the radius and the midpoint formula to find the center of a circle.

Core Standards Focus:
G-GPE Expressing Geometric Properties with Equations
Translate between the geometric description and the equation for a conic section

G-GPE.1 Derive the equation of a circle of given center and radius using the Pythagorean Theorem; complete the square to find the center and radius of a circle given by an equation.

Standards for Mathematical Practice:

SMP 1 – Make sense of problems and persevere in solving them

SMP 6 – Attend to precision

The Teaching Cycle:

Launch (Whole Class):
Begin the task by telling students that they will be solving the circle challenges by using the information given, ideas that they have learned in the past (like the distance and midpoint formulas), and their logic to write equations and justify conclusions about circles. It will probably be useful to have graph paper available to sketch the circles based on the information given.

Explore (Small Group):
Monitor students as they work, focusing on how they are making sense of the problems and using the information. Encourage students to draw the situation and visualize the circle to help when they are stuck. Insist upon justification, asking, "How do you know?"

Discuss (Whole Class):

Select problems that were challenging for the class or highlighted important ideas or useful strategies. Problem #4 is recommended for this purpose, but it is also important to select the problems that have generated interest in the class.

Aligned Ready, Set, Go: Connecting Algebra & Geometry 6.6

READY, SET, GO! Name Period Date

READY

Topic: Finding the distance between two points

Simplify. Use the distance formula $d=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$ **to find the distance between the given points. Leave your answer in simplest radical form.**

1. $A(18,-12)\ B(10,4)$
2. $G(-11,-9)\ H(-3,7)$
3. $J(14,-20)\ K(5,5)$
4. $M(1,3)\ P(-2,7)$
5. $Q(8,2)\ R(3,7)$
6. $S\left(-11,2\sqrt{2}\right)\ T\left(-5,-4\sqrt{2}\right)$
7. $W\left(-12,-2\sqrt{2}\right)\ Z\left(-7,-3\sqrt{2}\right)$

SET

Topic: Writing equations of circles

Use the information provided to write the equation of the circle in standard form,

$$(\boldsymbol{x}-\boldsymbol{h})^2+(\boldsymbol{y}-\boldsymbol{h})^2=\boldsymbol{r}^2$$

8. Center $(-16,-5)$ and the circumference is 22π
9. Center $(13,-27)$ and the area is 196π
10. Diameter measures 15 units and the center is at the intersection of y = x + 7 and y = 2x – 5
11. Lies in quadrant 2 Tangent to $x=-12$ *and* $x=-4$

12. Center (-14 , 9) Point on circle (1 , 11)

13. Center lies on the y axis Tangent to y = -2 and y = -17

14. Three points on the circle are $(-8,5),(3,-6),(14,5)$

15. I know three points on the circle are (-7,6), (9,6), and (-4,13). I think that the equation of the circle is $(x-1)^2+(y-6)^2=64$. Is this the correct equation for the circle? Justify your answer.

GO

Topic: Finding the value of ***B*** in a quadratic in the form of Ax^2+Bx+C in order to create a perfect square trinomial.

Find the value of *B* that will make a perfect square trinomial. Then write the trinomial in factored form.

16. $x^2+\_\_\_\_\_x+36$

17. $x^2+\_\_\_\_\_x+100$

18. $x^2+\_\_\_\_\_x+225$

19. $9x^2+\_\_\_\_\_x+225$

20. $16x^2+\_\_\_\_\_x+169$

21. $x^2+\_\_\_\_\_x+5$

22. $x^2+\_\_\_\_\_x+\frac{25}{4}$

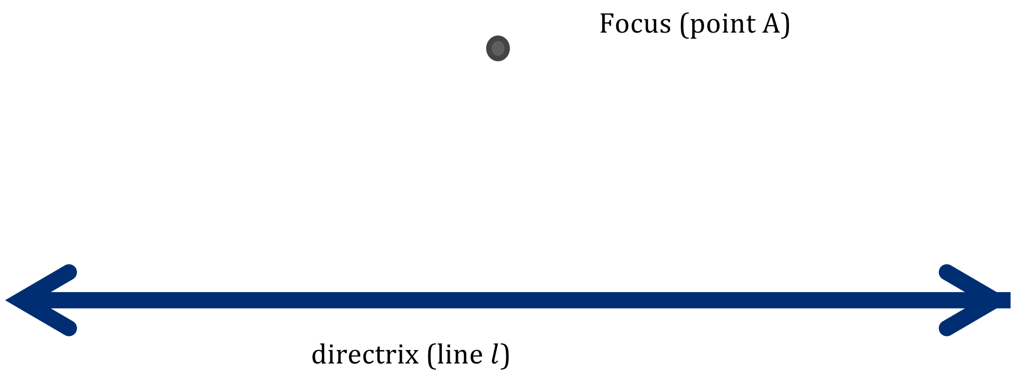
23. $x^2+\_\_\_\_\_x+\frac{9}{4}$

24. $x^2+\_\_\_\_\_x+\frac{49}{4}$

6.7 Directing Our Focus

A Develop Understanding Task

On a board in your classroom, your teacher has set up a point and a line like this:

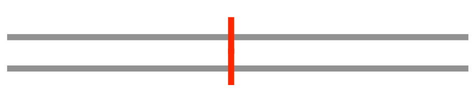


We're going to call the line a directrix and the point a focus. They've been labeled on the drawing.

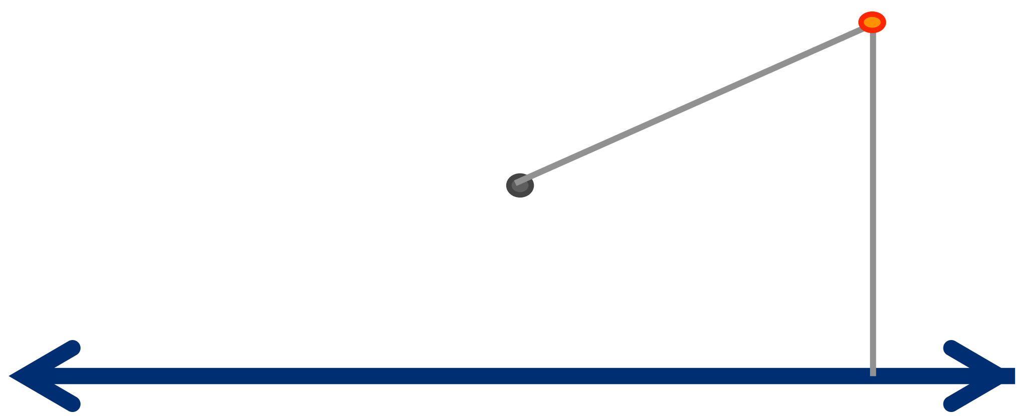
Similar to the circles task, the class is going to construct a geometric figure using the focus (point A) and directrix (line l).

1. Cut two pieces of string with the same length.

2. Mark the midpoint of each piece of string with a marker.



3. Position the string on the board so that the midpoint is equidistant from the focus (point A) and the directrix (line l), which means that it must be perpendicular to the directrix. While holding the string in this position, put a pin through the midpoint. Depending on the size of your string, it will look something like this:



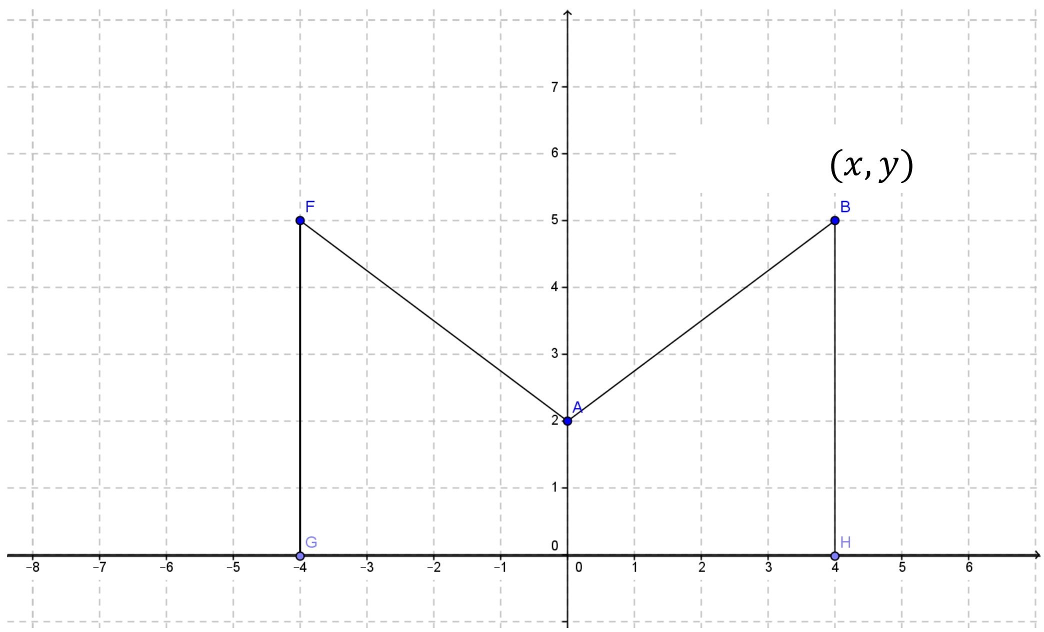
4. Using your second string, use the same procedure to post a pin on the other side of the focus.

5. As your classmates post their strings, what geometric figure do you predict will be made by the tacks (the collection of all points like (x, y) show in the figure above)? Why?

6. Where is the vertex of the figure located? How do you know?

7. Where is the line of symmetry located? How do you know?

8. Consider the following construction with focus point A and the x-axis as the directrix. Use a ruler to complete the construction of the parabola in the same way that the class constructed the parabola with string.



9. You have just constructed a parabola based upon the definition: A parabola is the set of all points (x, y) equidistant from a line l (the directrix) and a point not on the line (the focus). Use this definition to write the equation of the parabola above, using the point (x, y) to represent any point on the parabola.

10. How would the parabola change if the focus was moved up, away from the directrix?

11. How would the parabola change if the focus were to be moved down, toward the directrix?

12. How would the parabola change if the focus were to be moved down, below the directrix?

6.7 Directing Our Focus – Teacher Notes

A Develop Understanding Task

Purpose:

The purpose of this task is to develop the definition of a parabola as the set of all points equidistant from a given point (the focus) and a line (the directrix). Only those parabolas with horizontal directrices are considered in this task. Students develop an equation for a parabola based on the definition, using the distance formula. Students are also asked to consider the relationship between the focus and directrix and how the parabola changes as they are moved in relation to each other.

Core Standards Focus:

G.GPE Expressing Geometric Properties with Equations

Translate between the geometric description and the equation for a conic section

G.GPE.2. Derive the equation of a parabola given a focus and directrix.

Note: *Connect the equations of circles and parabolas to prior work with quadratic equations. The directrix should be parallel to a coordinate axis.*

Standards for Mathematical Practice:

SMP 7 – Look for and make use of structure

SMP 8 – Look for express regularity in repeated reasoning

The Teaching Cycle:

Launch (Whole Class):

Be prepared for the class activity by having scissors, markers, rulers, and string for students to use. Have a large corkboard with focus and directrix set up for students to use, as pictured in the task. Lead the class in following the directions for cutting and marking the strings and then posting them on the board. Before anyone posts a string, ask students what shape they think will be made and

why. Watch as students post their strings to be sure that they are perpendicular to the directrix and pulled tight both directions so that they look like the illustration.
After students have identified that the figure formed is a parabola, have them work individually on completing the diagram in #8. When completed, ask how they find the vertex point on a parabola? Be sure that the discussion includes the fact that the vertex will be the point on the line of symmetry that is the midpoint between the focus and the directrix. How is the vertex like other points on the parabola? (It is equidistant from the focus and the directrix.) How is it different? (It's the only point of the parabola on the line of symmetry.) Direct the discussion to the line of symmetry. Where is it on the parabola they just made? How could it be found on any parabola, given the focus and directrix?

After their work on #8, explain the geometric definition of a parabola given in #9. Then have students work together to use the definition to write the equation of the parabola.

Explore (Small Group):
Monitor students as they work to be sure that they are using the point marked (x, y) to represent any point on the parabola, rather than naming it (4,5). If they have written the equation using (4,5) then ask them how they would change their initial equation to call the point (x, y) instead. After they have written their equation they may want to test it with the point (4, 5) since they know it is on the parabola. If students need help getting started, help them to focus on the distance between the (x, y) and the focus (0,2) and (x, y) and the directrix, $y = 0$. Ask how they could represent those distances algebraically.

Be sure that students have time to share their ideas about problems 10 -12 so that the class discussion of the relationship of the focus and the directrix is robust.

Discuss (Whole Class):
When students have finished their work on the equation, ask a group to present and explain their work. A possible version is below:

Distance from (x, y) to focus $(0, 2)$ = distance from (x, y) to x-axis

$\sqrt{(x-0)^2+(y-2)^2}$	=	y	
$(x-0)^2+(y-2)^2$	=	y^2	Squaring both sides
x^2+y^2-4y+4	=	y^2	Simplifying
x^2-4y+4	=	0	Simplifying
x^2+4	=	$4y$	Solving for y
$\frac{x^2}{4}+1$	=	y	Solving for y

Ask students how this equation matches what they already know about the parabola they have drawn. Where is the vertex in the equation? How could they use the equation to predict how wide or narrow the parabola will be?

Turn the discussion to questions 10-12. Ask various students to explain their answers. Use the parabola applet to test their conjectures about the effect of moving the focus in relation to the directrix.

Aligned Ready, Set, Go: Connecting Algebra & Geometry 6.7

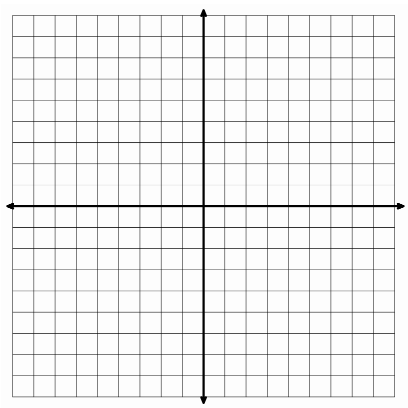
READY, SET, GO! Name Period Date

READY

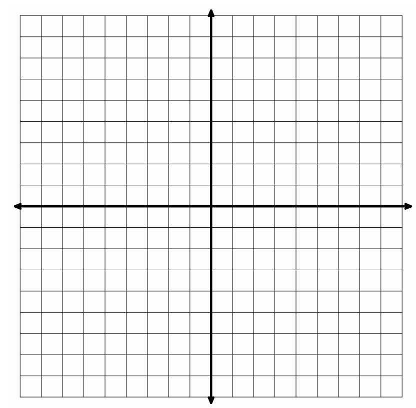
Topic: Graphing Quadratics

Graph each set of functions on the same coordinate axes. Describe in what way the graphs are the same and in what way they are different.

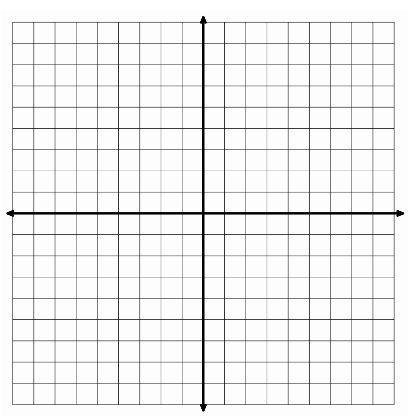
1. $y = x^2, y = 2x^2, y = 4x^2$



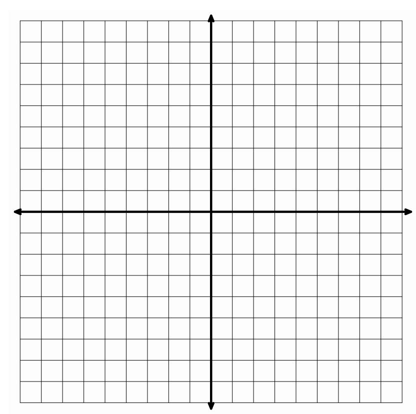
2. $y = \frac{1}{4}x^2, y = -x^2, y = -4x^2$



3. $y = \frac{1}{4}x^2, y = x^2 - 2, y = \frac{1}{4}x^2 - 2, y = 4x^2 - 2$



4. $y = x^2, y = -x^2, y = x^2 + 2, y = -x^2 + 2$



SET

Topic: Sketching a parabola using the conic definition.

Use the conic definition of a parabola to sketch a parabola defined by the given focus *F* and the equation of the directrix.

Begin by graphing the focus, the directrix, and point P_1. Use the distance formula to find FP_1 and find the vertical distance between P_1 and the directrix by identifying point H on the directrix and counting the distance. Locate the point P_2, (the point on the parabola that is a reflection of P_1 across the axis of symmetry.) Locate the vertex V. Since the vertex is a point on the parabola, it must lie equidistant between the focus and the directrix. Sketch the parabola. Hint: the parabola always "hugs" the focus.

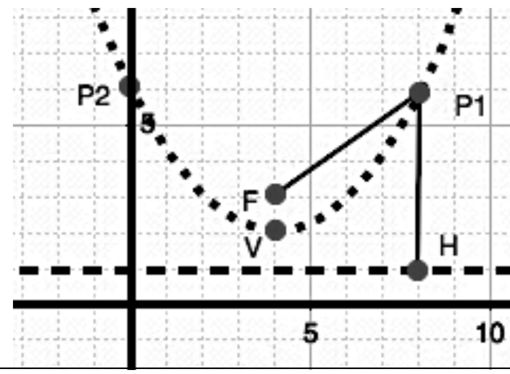
Example: ***F*** (4,3), ***P₁*** (8, 6), y = 1

FP₁ **=** $\sqrt{(4-8)^2+(3-6)^2}=\sqrt{16+9}=\sqrt{25}=5$

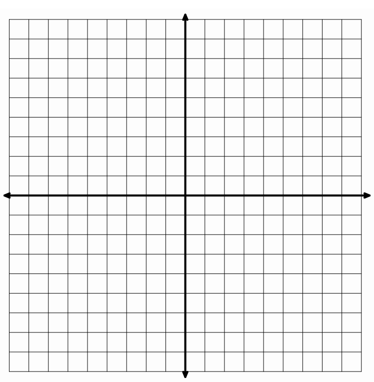
P₁H *= 5*

P₂ is located at (0, 6)

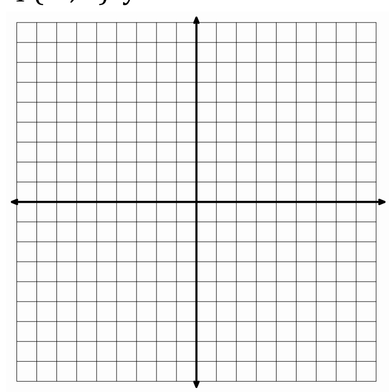
V is located at (4, 2)



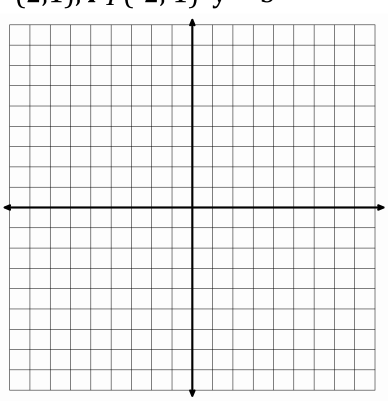
5. ***F*** (1,-1), ***P₁*** (3, -1) y = -3



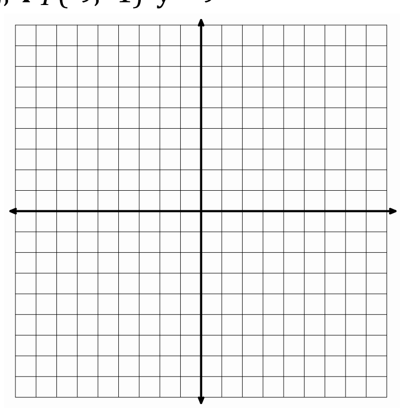
6. ***F*** (-5,3), ***P₁*** (-1, 3) y = 7



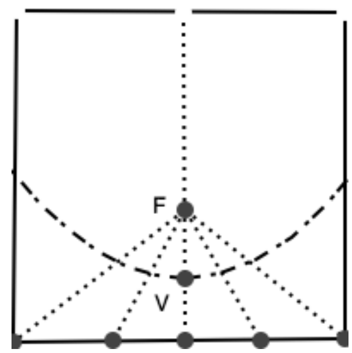
7. ***F*** (2,1), ***P₁*** (-2, 1) y =-3



8. ***F*** (1, -1), ***P₁*** (-9, -1) y = 9



9. Find a square piece of paper (a post-it note will work). Fold the square in half vertically and put a dot anywhere on the fold. Let the edge of the paper be the directrix and the dot be the focus. Fold the edge of the paper (the directrix) up to the dot repeatedly from different points along the edge. The fold lines between the focus and the edge should make a parabola.



Experiment with a new paper and move the focus.
Use your experiments to answer the following questions.

10. How would the parabola change if the focus were moved up, away from the directrix?

11. How would the parabola change if the focus were moved down, toward the directrix?

12. How would the parabola change if the focus were moved down, below the directrix?

GO

Topic: Finding the center and radius of a circle.

Write each equation so that it shows the center (h, k) and radius r of the circle. This called the standard form of a circle. $(x-h)^2+(y-k)^2=r^2$

13. $x^2+y^2+4y-12=0$

14. $x^2+y^2-6x-3=0$

15. $x^2+y^2+8x+4y-5=0$

16. $x^2+y^2-6x-10y-2=0$

17. $x^2+y^2-6y-7=0$

18. $x^2+y^2-4x+8y+6=0$

19. $x^2+y^2-4x+6y-72=0$

20. $x^2+y^2+12x+6y-59=0$

21. $x^2+y^2-2x+10y+21=0$

22. $4x^2+4y^2+4x-4y-1=0$

6.8 Functioning With Parabolas

A Solidify Understanding Task

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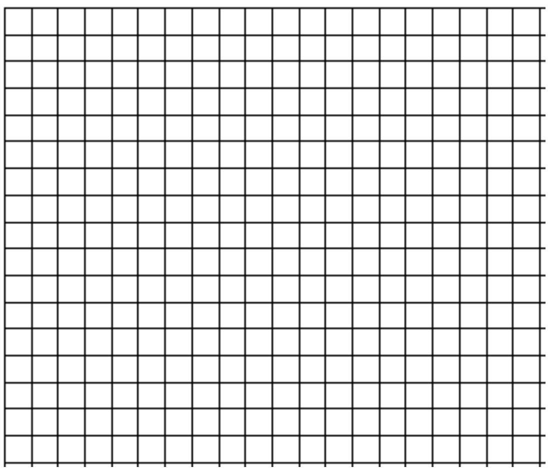
Sketch the graph (accurately), find the vertex and use the geometric definition of a parabola to find the equation of these parabolas.

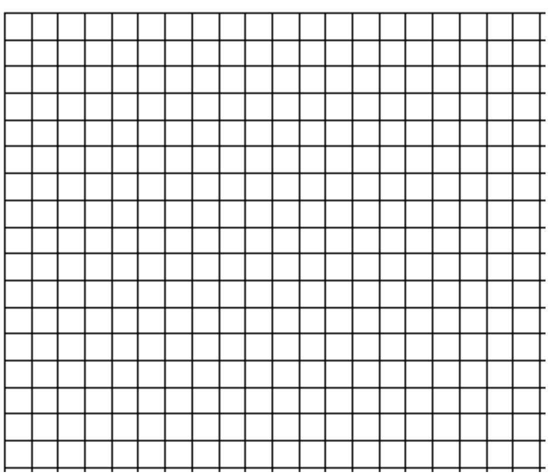
1. Directrix $y = -4$, Focus A(2, -2)



Vertex ________________

Equation:

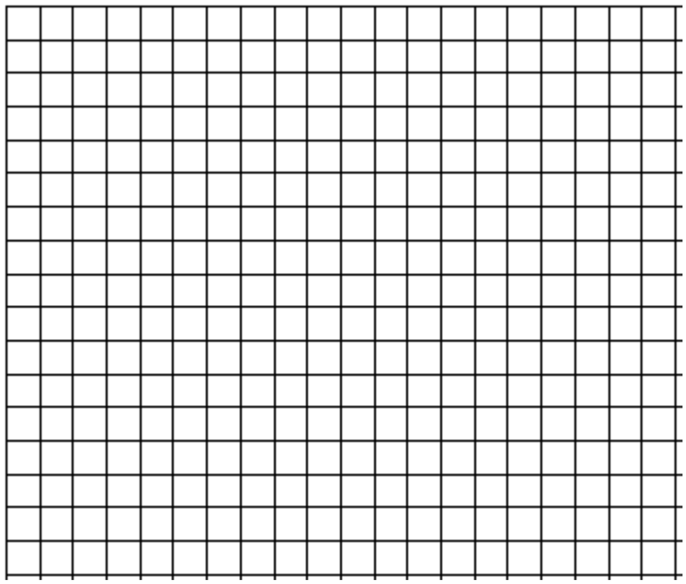
2. Directrix $y = 2$, Focus A(-1, 0)



Vertex ________________

Equation:

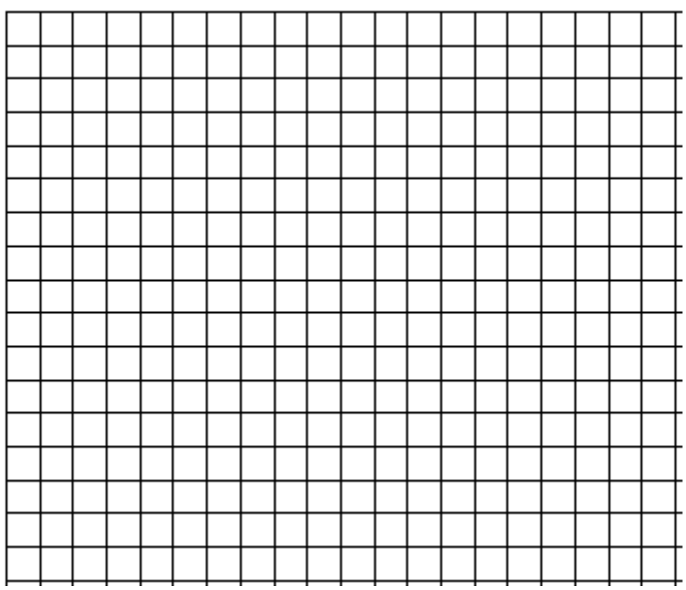
3. Directrix $y = 3$, Focus A(1, 7)



Vertex ________________

Equation:

3. Directrix $y = 3$, Focus A(2, -1)



Vertex ________________

Equation:

4. Given the focus and directrix, how can you find the vertex of the parabola?

5. Given the focus and directrix, how can you tell if the parabola opens up or down?

6. How do you see the distance between the focus and the vertex (or the vertex and the directrix) showing up in the equations that you have written?

7. Describe a pattern for writing the equation of a parabola given the focus and directrix.

8. Annika wonders why we are suddenly thinking about parabolas in a completely different way than when we did quadratic functions. She wonders how these different ways of thinking match up. For instance, when we talked about quadratic functions earlier we started with $y = x^2$. "Hmmmm. .... I wonder where the focus and directrix would be on this function," she thought. Help Annika find the focus and directrix for $y = x^2$.

9. Annika thinks, "Ok, I can see that you can find the focus and directrix for a quadratic function, but what about these new parabolas. Are they quadratic functions? When we work with families of functions, they are defined by their rates of change. For instance, we can tell a linear function because it has a constant rate of change." How would you answer Annika? Are these new parabolas quadratic functions? Justify your answer using several representations and the parabolas in problems 1-4 as examples.

6.8 Functioning With Parabolas

A Solidify Understanding Task

Purpose: The purpose of this task is to solidify students' understanding of the geometric definition of a parabola and to connect it to their previous experiences with quadratic functions. The task begins with students writing equations for specific parabolas with specific relationships between the focus and directrix. Students use this experience to generalize a strategy for writing the equation of a parabola, solidifying how to find the vertex and to use the distance between the focus and the vertex (or the distance between the vertex and the directrix) in writing an equation. Students are then asked to find the focus and directrix for $y = x^2$ to illustrate that the focus and directrix could be identified for the parabolas that they worked with as the graphs of quadratic functions. Finally, they are asked to verify that parabolas constructed with a horizontal directrix from a geometric perspective will also be quadratic functions, based upon a linear rate of change.

Core Standards Focus:

G.GPE Expressing Geometric Properties with Equations

Translate between the geometric description and the equation for a conic section

G-GPE.2. Derive the equation of a parabola given a focus and directrix.

Note: *Connect the equations of circles and parabolas to prior work with quadratic equations. The directrix should be parallel to a coordinate axis.*

Standards for Mathematical Practice:

SMP 8 – Look for express regularity in repeated reasoning

The Teaching Cycle:

Launch (Whole Class):

Begin by having students individually work the first problem. Have one student that has done a good job of accurately sketching the parabola demonstrate for the class. The first problems are very similar to the work done in "Directing Our Focus", but each problem has been selected so that students will see different distances between the focus and the directrix and use them to draw conclusions later in the task. After the first problem is done as a class, the rest of the task can be done in small groups.

Explore (Small Group):

As students are working on the task, listen to see what they are noticing about finding the vertex. They should identify that the vertex is on the line of symmetry, which is perpendicular to the directrix, and that the vertex is the midpoint between the focus and directrix. They should also be noticing how it shows up in the equation, particularly that it is easier to recognize if the $(x-h)^2$ term in the equation is not expanded. They should also notice the distance from the vertex to the focus, *p,* and where that is occurring in the equation. Identify students for the discussion that can describe the patterns that they see with the parabola and the equation and have developed a good "recipe" for writing an equation.

As you monitor student work on #10, identify student use of tables, equations, and graphs to demonstrate that the parabolas they are working with fit into the quadratic family of functions because they have linear rates of change.

Discuss (Whole Class):

Begin the discussion with question #8. Ask a couple of groups that have developed an efficient strategy for writing the equation of a parabola given the focus and directrix to present their work. (Students will be asked to generate a general form of the equation in the RSG). Ask the class to compare and edit the strategies so that they have a method that they are comfortable with using for this purpose. Then ask them to use the process in reverse and tell how they found the focus and directrix for $y = x^2$ (question 9).

Move the discussion to #10. Ask various students to show how the parabolas are quadratic functions using tables, graphs, and equations. Focus on how the linear rate of change shows up in each representation. Connect the equations and graphs to the transformation perspective that they worked with in previous modules.

Aligned Ready, Set, Go: Connecting Algebra & Geometry 6.8

READY, SET, GO! Name Period Date

READY

Topic: Standard form of a quadratic.

Verify that the given point lies on the graph of the parabola described by the equation. (Show your work.)

1. $(6,0)$ $y=2x^2-9x-18$

2. $(-2,49)$ $y=25x^2+30x+9$

3. $(5,53)$ $y=3x^2-4x-2$

4. $(8,2)$ $y=\frac{1}{4}x^2-x-6$

SET

Topic: Equation of parabola based on the geometric definition

5. Verify that $(y-1)=\frac{1}{4}x^2$ is the equation of the parabola in *figure 1* by plugging in the 3 points V (0,1), C (4,5) and E (2,2). Show your work for each point!

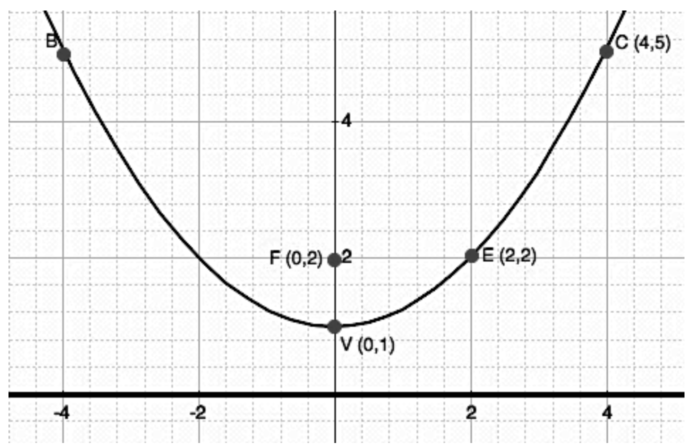
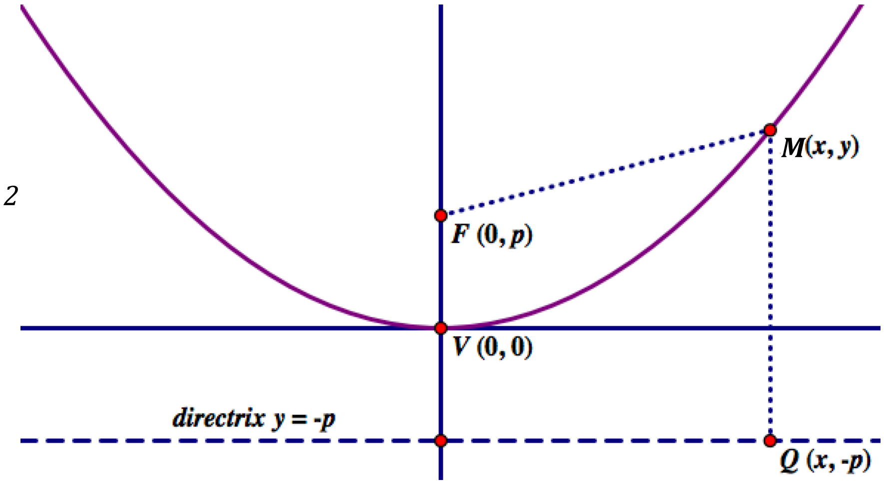


Figure 1

6. If you didn't know that (0,1) was the vertex of the parabola, could you have found it by just looking at the equation? Explain.

7. Use the diagram in *figure 2* to derive the general equation of a parabola based on the **geometric definition** of a parabola. Remember that the definition states that MF = MQ.

Figure 2

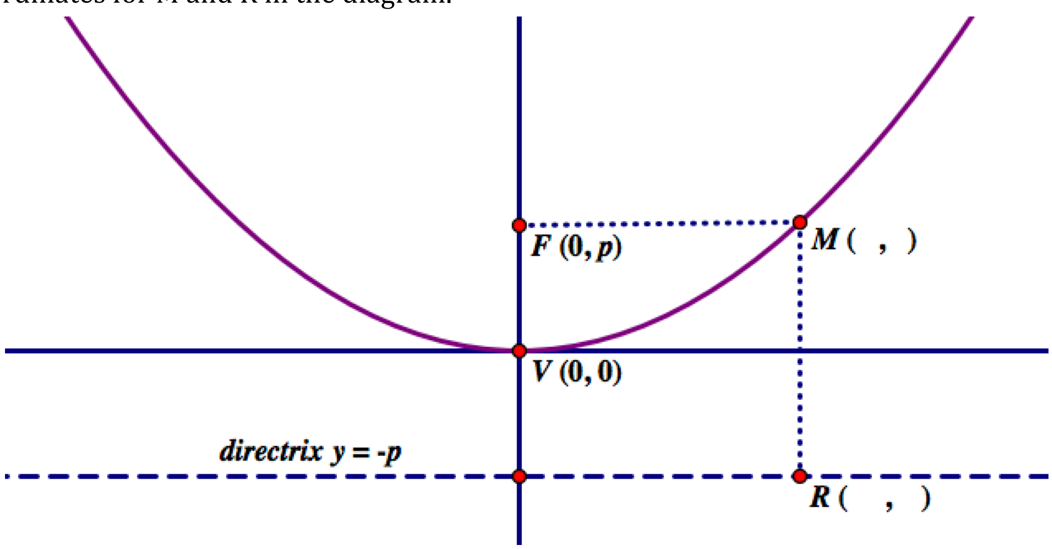


8. Recall the equation in #5, $(y-1)=\frac{1}{4}x^2$, what is the value of $\boldsymbol{p}$?

9. In general, what is the value of $\boldsymbol{p}$ for any parabola?

10. In *figure 3*, the point M is the same height as the focus and $\overline{FM} \cong \overline{MR}$. How do the coordinates of this point compare with the coordinates of the focus?
Fill in the missing coordinates for M and R in the diagram.

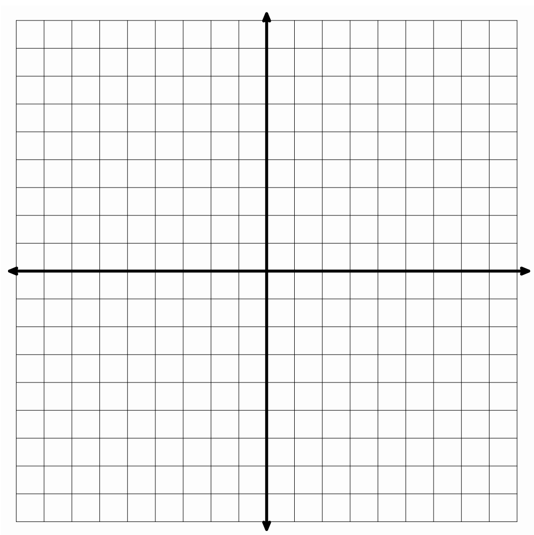
Figure 3



Sketch the graph by finding the vertex and the point M and R (the reflection of M) as defined in the diagram above. Use the geometric definition of a parabola to find the equation of these parabolas.

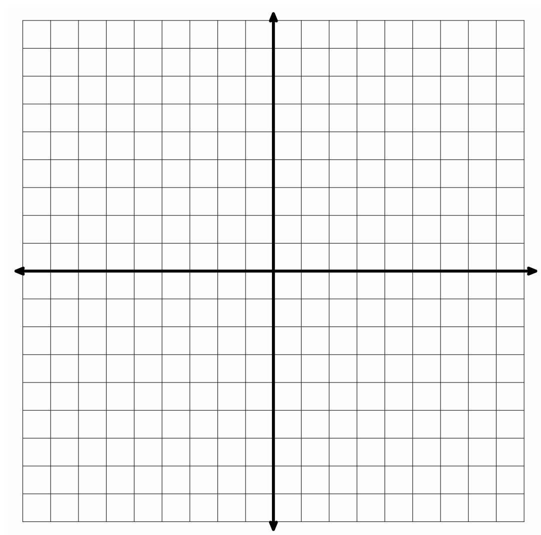
11. Directrix $y = 9$, Focus $F(-3, 7)$

Vertex ___________
Equation ___________________________________



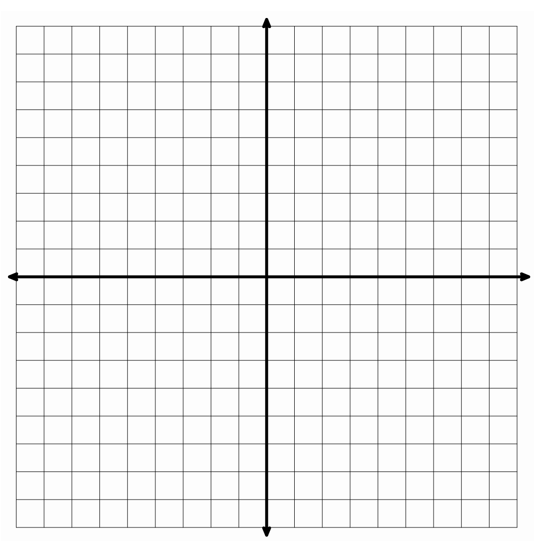
12. Directrix $y = -6$, Focus $F(2, -2)$

Vertex ___________
Equation ___________________________________



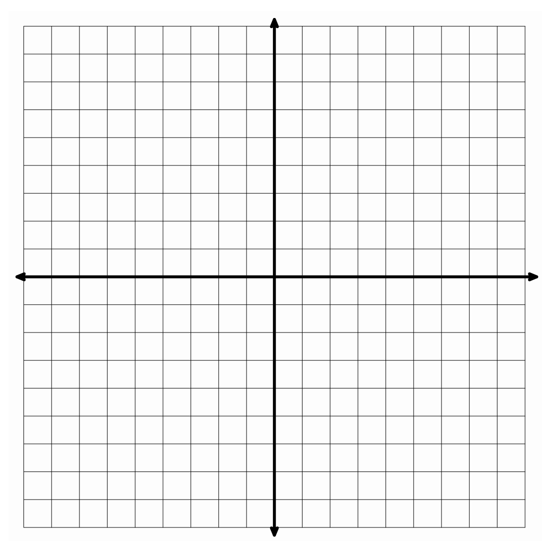
13. Directrix $y = 5$, Focus $F(-4, -1)$

Vertex ___________
Equation ___________________________________



14. Directrix $y = -1$, Focus $F(4, -3)$

Vertex ___________
Equation ___________________________________



GO

Topic: Finding minimum and minimum values for quadratics

Find the maximum or minimum value of the quadratic. Indicate which it is.

15. $y = x^2 + 6x - 5$

16. $y = 3x^2 - 12x + 8$

17. $y = -\frac{1}{2}x^2 + 10x + 13$

18. $y = -5x^2 + 20x - 11$

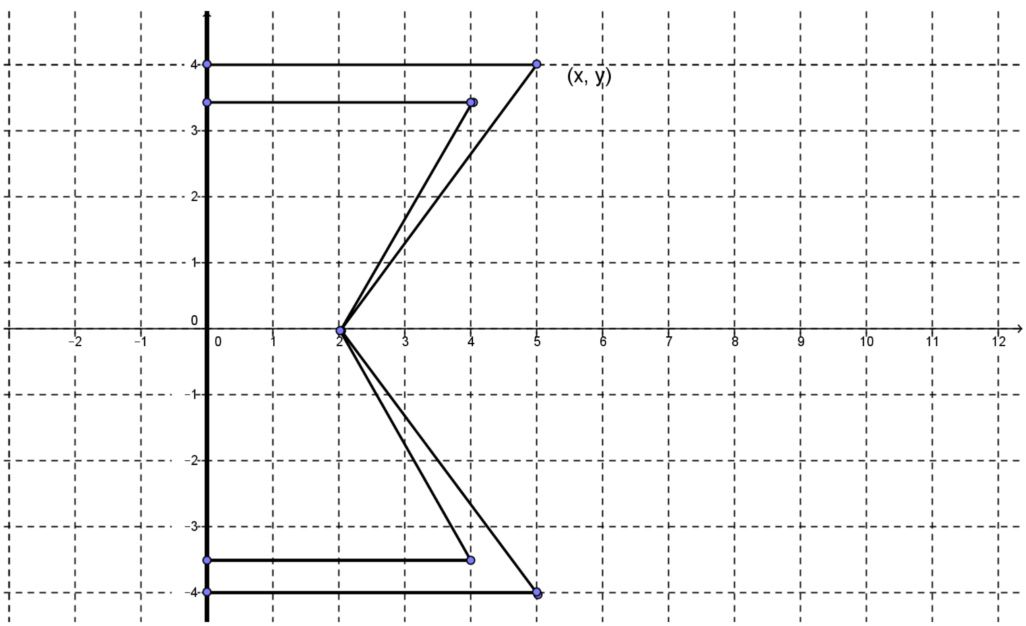
19. $y = \frac{7}{2}x^2 - 21x - 3$

20. $y = -\frac{3}{2}x^2 + 9x + 25$

6.9 Turn It Around

A Solidify Understanding Task

Annika is thinking more about the geometric view of parabolas that she has been working on in math class. She thinks, "Now I see how all the parabolas that come from graphing quadratic functions could also come from a given focus and directrix. I notice that all the parabolas have opened up or down when the directrix is horizontal. I wonder what would happen if I rotated the focus and directrix 90 degrees so that the directrix is vertical. How would that look? What would the equation be? Hmmm...." So Annika starts trying to construct a parabola with a vertical directrix. Here's the beginning of her drawing. Use a ruler to complete Annika's drawing.



1. Use the definition of a parabola to write the equation of Annika's parabola.

2. What similarities do you see to the equations of parabolas that open up or down? What differences do you see?

3. Try another one: Write the equation of the parabola with directrix $x = 4$ and focus (0, 3).

4. One more for good measure: Write the equation of the parabola with directrix $x = -3$ and focus (-2, -5).

5. How can you predict if a parabola will open left, right, up, or down?

6. How can you tell how wide or narrow a parabola is?

7. Annika has two big questions left. Write and explain your answers to these questions.
   a. Are all parabolas functions?

   b. Are all parabolas similar?

6.9 Turn It Around – Teacher Notes

A Solidify Understanding Task

Special Note to Teachers: Rulers should be available for student use in this task.

Purpose: The purpose of this task is to generalize the work that students have done with parabolas that have a horizontal directrix (including those generated as quadratic functions), and extend the idea to parabolas with a vertical directrix. In the task, they graph and write equations for parabolas that have vertical directrices. They are asked to consider the idea that not all parabolas are functions, even though they have quadratic equations. The task ends with constructing an argument that all parabolas, like circles, are similar.

Core Standards Focus:

G.GPE Expressing Geometric Properties with Equations

Translate between the geometric description and the equation for a conic section

G.GPE.2. Derive the equation of a parabola given a focus and directrix.

Note: Connect the equations of circles and parabolas to prior work with quadratic equations. The directrix should be parallel to a coordinate axis.

Standards for Mathematical Practice:

SMP 7 – Look for and make use of structure

SMP 8 – Look for express regularity in repeated reasoning

The Teaching Cycle:

Launch (Whole Class):

Before handing out the task, ask students to think back to the lesson when they constructed a parabola by placing tacks on a board with a given focus and horizontal directrix. Ask students what shape would be constructed if they did the same thing with the strings and tacks, but the directrix was vertical and the focus was to the right of the directrix. After a brief discussion, distribute the task and have students complete the diagram and write the equation of the parabola. Ask a student to demonstrate how they wrote the equation using the distance formulas, just like they did previously with other parabolas. After the demonstration, students can work together to discuss the remaining questions in the task.

Explore (Small Group):
Monitor student work as they write the equations to see that they are considering which expressions to expand and simplify. Since they have previously expanded the y^2 expression, they may not recognize that it will be more convenient in this case to expand the $(x - b)^2$ term.

Listen to student discussion of #7 to find productive comments for the class discussion. Students should be talking about the idea that a function has exactly one output for each input, unlike these parabolas. Some may also talk about the vertical line test. Encourage them to explain the basis for the vertical line test, rather than just to cite it as a rule.

The question about whether all parabolas are similar may be more controversial because they don't seem to look similar in the way that other shapes do. Listen to students that are reasoning using the ideas of translation and dilation, particularly noting how they can justify this using a geometric perspective with the definition or arguing from the equation.

Discuss (Whole Class):
Begin the discussion with question #5. Press students to explain how to tell which direction the parabola opens given an equation or focus and directrix. Create a chart that solidifies the conclusions for students.

Move the discussion to question #7a. Ask students to describe why some parabolas are not functions. Be sure that the discussion relies on the idea of a function having exactly one output for each input, rather than simply the vertical line test or the idea that it's not a function if the equation contains a y^2. In either case, press students to relate their idea to the definition of function.

Close the discussion with students' ideas about question #7b. Allow the arguments to be informal, but focused on how they know that any parabola can be obtained from any other by the process of dilation and translation.

Aligned Ready, Set, Go: Connecting Algebra & Geometry 6.9

READY, SET, GO! Name Period Date

READY

Topic: Circles Review

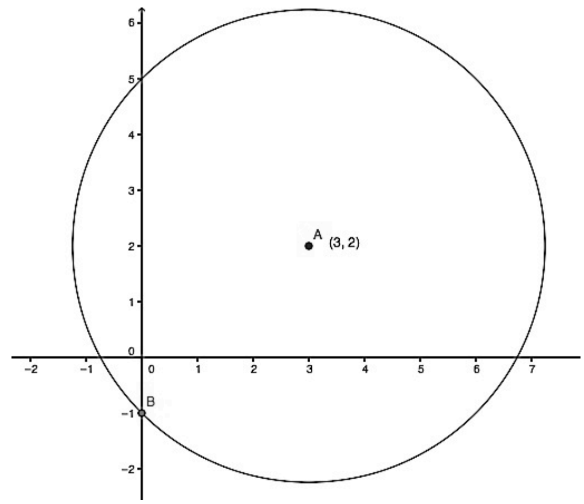
Use the given information to write the equation of the circle in standard form.

1. Center: (-5, -8) , Radius: 11

2. Endpoints of the diameter: (6, 0) and (2, -4)

3. Center (-5, 4) : Point on the circle (-9, 1)

4. Equation of the circle in the diagram to the right.

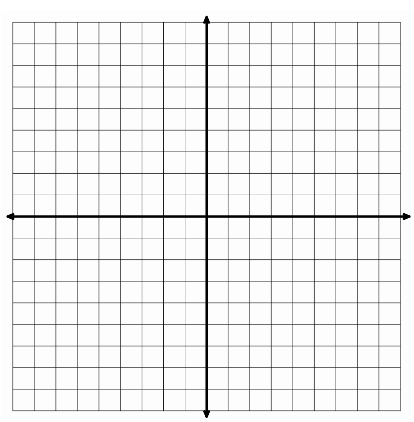


SET

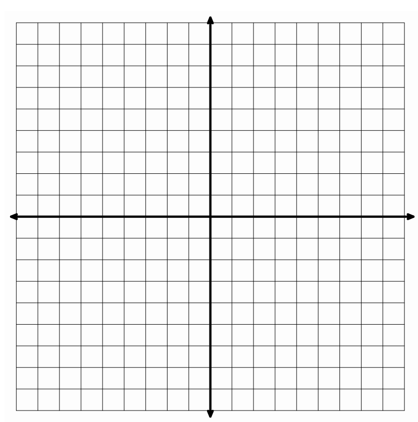
Topic: Writing equations of horizontal parabolas.

Use the focus F, point M, a point on the parabola, and the equation of the directrix to sketch the parabola (label your points) and write the equation. Put your equation in the form $x = \frac{1}{4p}(y-k)^2 + h$ where "p" is the distance from the focus to the vertex.

5. ***F*** (1,0), ***M*** (1,4) x = -3



6. ***F*** (3,1), ***M*** (3,-5) x = 9



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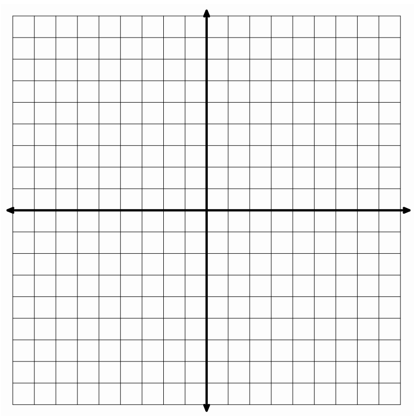
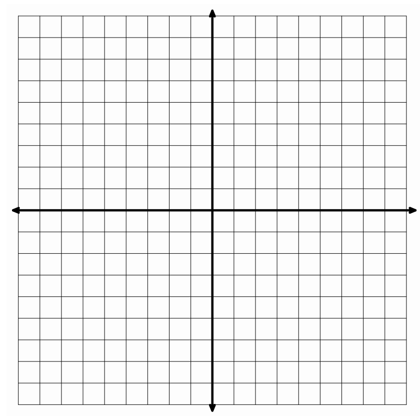
7. ***F*** (7,-5), ***M*** (4,-1) x = 9



8. ***F*** (-1,2), ***M*** (6,-9) x = -7



GO

Topic: Identifying key features of a quadratic written in vertex form

State (a) the coordinates of the vertex, (b) the equation of the axis of symmetry, (c) the domain, and (d) the range for each of the following functions.

9. $f(x)=(x-3)^2+5$

10. $f(x)=(x+1)^2-2$

11. $f(x)=-(x-3)^2-7$

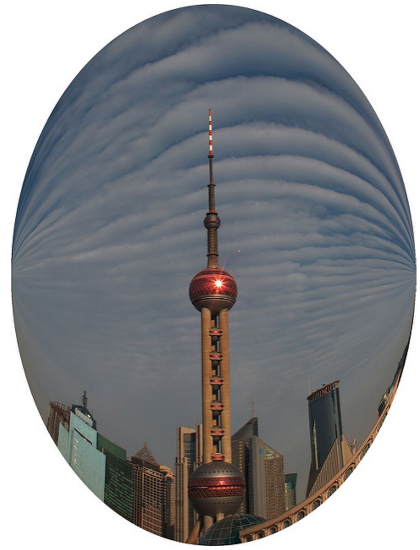
12. $f(x)=-3\left(x-\frac{3}{4}\right)^2+\frac{4}{5}$

13. $f(x)=\frac{1}{2}(x-4)^2+1$

14. $f(x)=\frac{1}{4}(x+2)^2-4$

15. Compare the vertex form of a quadratic to the geometric definition of a parabola based on the focus and directrix. Describe how they are similar and how they are different.

6.10H Operating On A Shoestring



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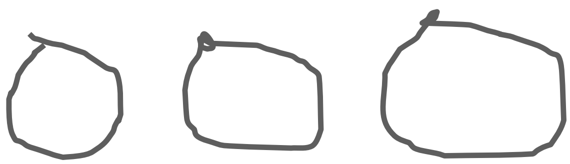
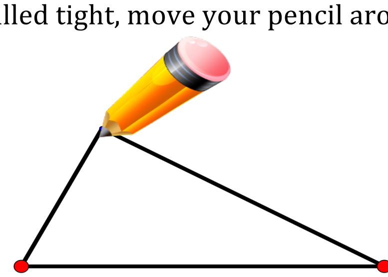
A Solidify Understanding Task

You will need 3 pieces of paper, a piece of cardboard that is at least 8" x 8", 2 tacks, 36 inches of string, and a pencil

1. Cut three pieces of string: a 10 inch piece, a 12 inch piece, and a 14 inch piece. Tie the ends of each piece of string together, making 3 loops.



2. Place a piece of paper on top of the cardboard.
3. Place the two tacks 4 inches apart, wrap the string around the tacks and then press the tacks down.
4. Pull the string tight between the two tacks and hold them down between your finger and thumb. Pull the string tight so that it forms a triangle, as shown below. What is the length of the part of the string that is not on the segment between the two tacks, the sum of the lengths of the segments marked d1 and d2 in the diagram?
5. With your pencil in the loop and the string pulled tight, move your pencil around the path that keeps the string tight.



6. What shape is formed? What geometric features of the figure do you notice?

7. Repeat the process again using the other strings. What is the effect of the length of the string?

8. What is the effect of changing the distance between the two tacks? (You may have to experiment to find this answer.)

The geometric figure that you have created is called an ellipse. The two tacks each represent a focus point for the ellipse. (The plural of the word "focus" is "foci", but "focuses" is also correct.)

To focus our observations about the ellipse, we're going to slow the process down and look at points on the ellipse in particular positions. To help make the labeling easier, we will place the ellipse on the coordinate plane.

9. The distances from the point on the ellipse to each of the two foci is labeled d1 and d2.

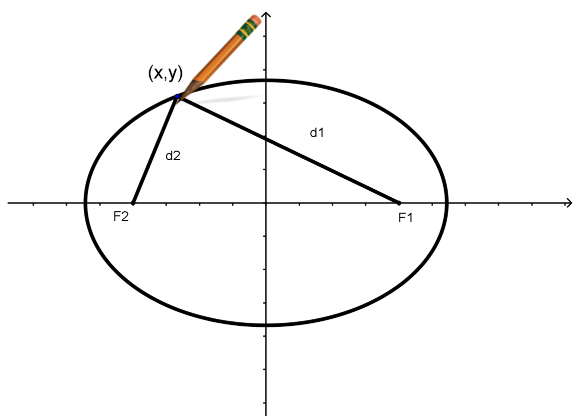


Figure 1

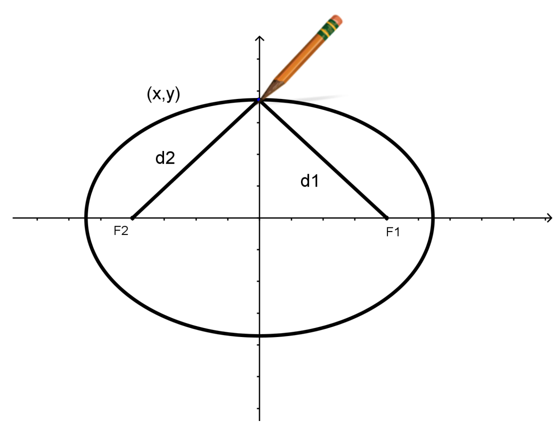
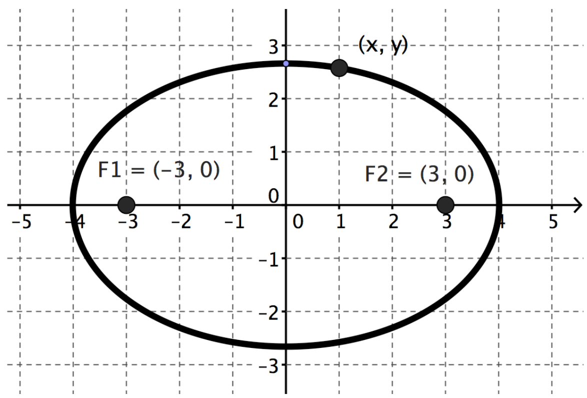


Figure 2

How does d1 + d2 in Figure 1 compare to d1 + d2 in Figure 2? (Figure 1 and Figure 2 are the same ellipse.)

10. How does d1 + d2 compare to the length of the ellipse, measured from one end to the other along the x-axis? Explain your answer with a diagram.

You have just constructed an ellipse based upon the definition: An ellipse is the set of all points *(x, y)* in a plane which have the same total distance from two fixed points called the foci. Like circles and parabolas, ellipses also have equations. The basic equation of the ellipse is derived in a way that is similar to the equation of a parabola or a circle. Since it's usually easier to start with a specific case and then generalize, we'll start with this ellipse:



11. Now, use the conclusions that you drew earlier to help you to write an equation. (We'll help with a few prompts.)
   a. What is the sum of the distances from a point (x, y) on this ellipse to F1 and F2?

   b. Write an expression for the distance between the point (x, y) on the ellipse and F1(-3,0).

   c. Write an expression for the distance between (x,y) on the ellipse and F2 (3,0).

   d. Use your answers to a, b, and c to write an equation.

12. The equation of this ellipse in standard form is:

$$\frac{x^2}{16} + \frac{y^2}{7} = 1$$

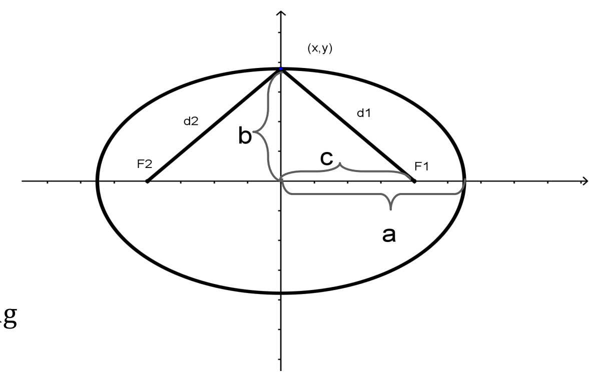
It might be much trickier than you would imagine to re-arrange your equation to check it, so we'll try a different strategy. This equation would say that the ellipse contains the points $(4,0)$ and $(0,-\sqrt{7})$. Do both of these points make your equation true? Show how you checked them here.

13. Using the standard form of the equation is actually pretty easy, but you have to notice a few more relationships. Here's another picture with some different parts labeled.

a = horizontal distance from the center to the ellipse

b = vertical distance from the center to the ellipse

c = distance from the center to a focus



Based on the diagram, describe in words the following expressions:

2a

2b

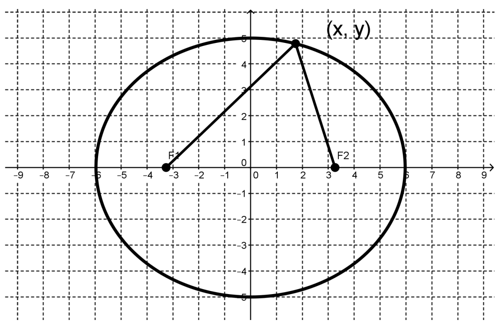
2c

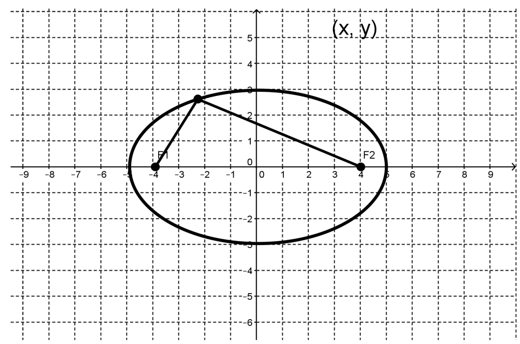
14. What is the mathematical relationship between a, b, and c?

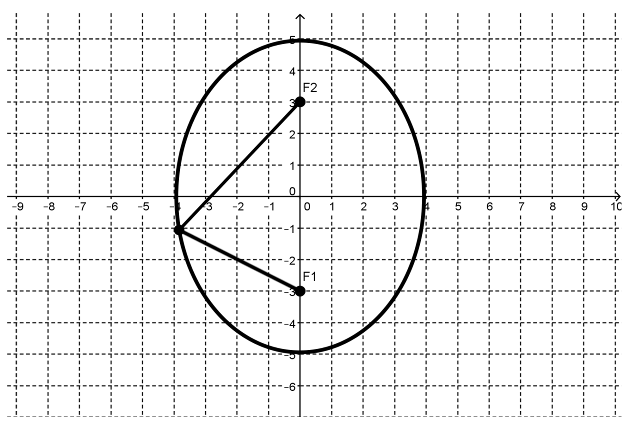
15. Now you can use the standard form of the equation of an ellipse centered at (0,0) which is:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

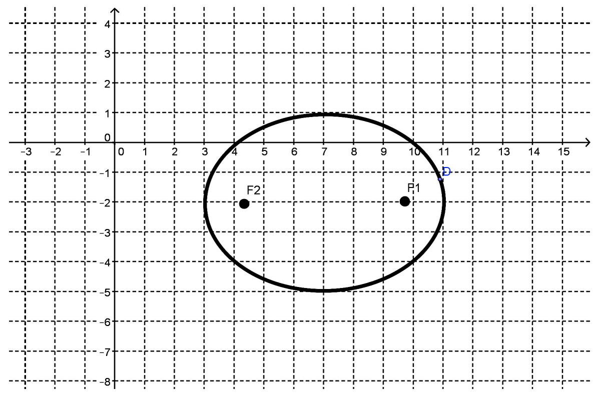
Write the equation of each of the ellipses pictured below:







16. Based on your experience with shifting circles and parabolas away from the origin, write an equation for of this ellipse. Test your equation with some points on the ellipse that you can identify.



6.10H Operating On A Shoestring – Teacher Notes

A Solidify Understanding Task

Note: Having a dynamic sketch of an ellipse available for the class discussion would be very helpful. One possibility is a free applet available at: http://www.cut-the-knot.org/Curriculum/Geometry/EllipseFocal.shtml.

Purpose:

The purpose of this task is to develop the definition of an ellipse as the set of all points in a plane such that the sum of the distances from a point on the ellipse to the two foci is constant. The task begins with having students construct ellipses with different lengths of strings, based upon the definition. They are asked to notice features of the ellipse such as the symmetries and the relationship between the length of the ellipse and the sum of the distances from a point to the foci, how the ellipse changes as the distance between the foci changes, and how the ellipse changes if the foci are changed from a horizontal axis to a vertical axis. Students are asked to use the definition to write the equation of a particular ellipse and introduced to standard form of the equation an ellipse. The task concludes with writing the equation for the graphs of several different ellipses.

Core Standards Focus:

G.GPE Expressing Geometric Properties with Equations

Translate between the geometric description and the equation for a conic section.

G.GPE.3 (+) Derive the equations of ellipses and hyperbolas given the foci, using the fact that the sum or difference of distances from the foci is constant.

Standards for Mathematical Practice:

SMP 7 – Look for and make use of structure

SMP 8 – Look for express regularity in repeated reasoning

The Teaching Cycle:

Launch (Whole Class):

Be prepared for the class activity by having scissors, markers, rulers, cardboard, and string for students to use.

Lead the class in following the directions for cutting and placing the strings to begin constructing the figure without actually demonstrating what shape will be formed. Watch as students draw their ellipses, keeping them focused on the part of the string that shows the distances between a given point on the ellipse and the foci. In this part of the activity, they should be noticing how the different string length affects the shape of the ellipse, and then how changing the distance between foci changes the ellipse for a given string length.

After students have had time to explore the ellipses and completed questions 1-8, lead a discussion of their observations. Some of the ideas that should come out:

- Increasing the length of the string without changing the distance between foci makes the ellipse longer.
- Increasing the distance between the foci without changing the length of the string makes the ellipse narrower, but not longer. Students may also notice that as the foci get closer, the ellipse gets rounder, or more like a circle. (This idea is addressed in the next task.)

At this point, students are ready to work on the rest of the task. Before beginning #9, be sure that they understand that d1 and d2 are the distances between the point and the two foci.

Explore (Small Group):

Much of the work in the remainder of the task depends upon making sense of #9 and #10, so monitor student work to ensure that each group works through the idea that the sum of d1 and d2 is constant and that d1 + d2 is the length of the string and the length of major axis of the ellipse. One way that they can see this is to use their string setup for the ellipse and look at d1 and d2 when the pencil point is on the end of the major axis.

As they begin to work through the algebra of writing the equation, you may need to remind them to use the distance formula to write the expressions that are needed to form the equation.

Discuss (Whole Class):

Begin the discussion with standard form of the equation of the ellipse for #12. Ask students how they could use this equation to describe the ellipse. Help them to notice that in the equation $\frac{x^2}{16}+\frac{y^2}{7}=1$, the horizontal length of the ellipse (the major axis) will be 8 and the vertical height of the ellipse (the minor axis) will be $2\sqrt{7}$.

Ask students to present their work in writing equations for each of the ellipses given in #15 and 16. If time permits, you may wish to give a few equations and ask students to quickly graph the equations just by using the length and height of the ellipse and sketching in the remainder.

Aligned Ready, Set, Go: Connecting Algebra & Geometry 6.10H

READY, SET, GO! Name Period Date

READY

Topic: Solving radical equation

Solve for x. Beware of *extraneous solutions.*

1. $\sqrt{2x-5}=3$

2. $\sqrt{10x+9}=13$

3. $\sqrt{2x}=x-4$

4. $3\sqrt{2x+2}=2\sqrt{5x-1}$

5. $x-3=\sqrt{3x+1}$

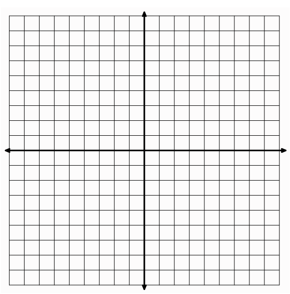
6. $4-\sqrt{10-3x}=x$

SET

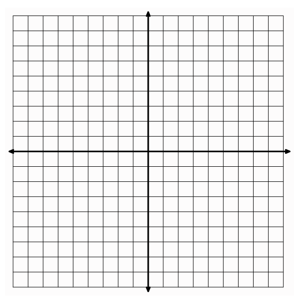
Topic: Graphing ellipses

Find the x- and y- intercepts of the ellipse whose equation is given. Then draw the graph.

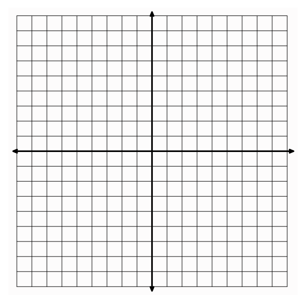
7. $\frac{x^2}{36}+\frac{y^2}{16}=1$



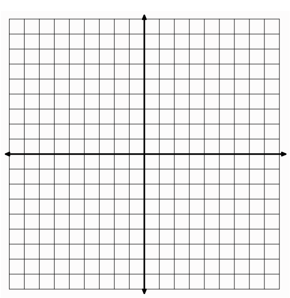
8. $\frac{x^2}{9}+\frac{y^2}{64}=1$



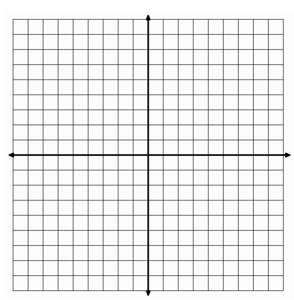
9. $\frac{x^2}{25}+\frac{y^2}{4}=1$



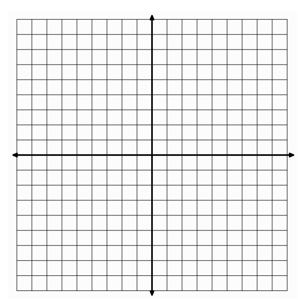
10. $x^2+4y^2=64$



11. $9x^2+y^2=36$



12. $x^2+3y^2=75$



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Not all ellipses are centered at the origin. An ellipse with center (h, k) is translated h units horizontally and k units vertically. The standard form of the equation of an ellipse with center at C (h, k) and whose vertices horizontally and vertically are ±a and ±b, respectively, is $\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$.

Write an equation, in standard form, for each ellipse based on the given center *C* and it's the given values for horizontal radius, *a*, and the vertical radius, *b*.

13. C (-2,3), a = ±4, b = ± 2

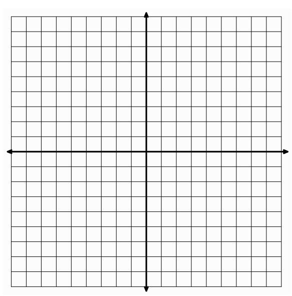
14. C (5,2), a = ±3, b = ± 5

15. C (-4,-7), a = ±10, b = ± 8

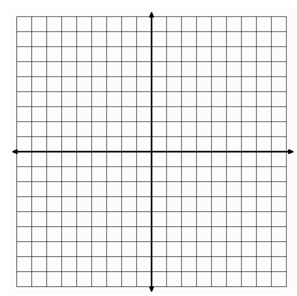
16. C (6,-5), a = ±7, b = ± $\sqrt{11}$

Write the equation of each ellipse in standard form. Identify the center. Then graph the ellipse.

17. $4x^2 + y^2 - 32x - 4y + 52 = 0$



18. $16x^2 + 9y^2 - 96x + 72y + 144 = 0$



GO

Topic: Point-slope form of a line.

The rectangle in figure B is a translation of the rectangle in figure A. Write the equations of the 2 diagonals of rectangle ABCD in point-slope form. Then write the equations of the 2 diagonals of A'B'C'D'.

19. *figure* A

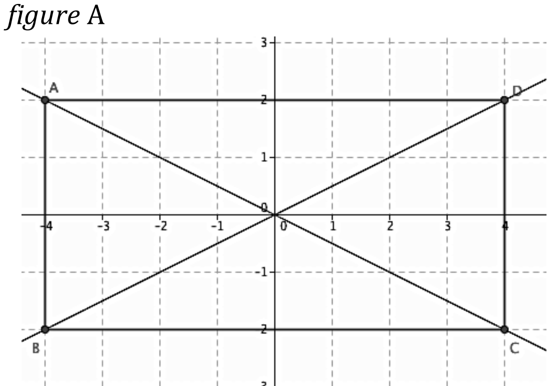
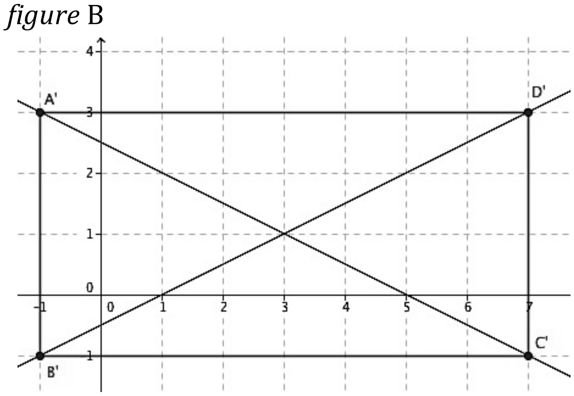


figure B



20. *figure* A

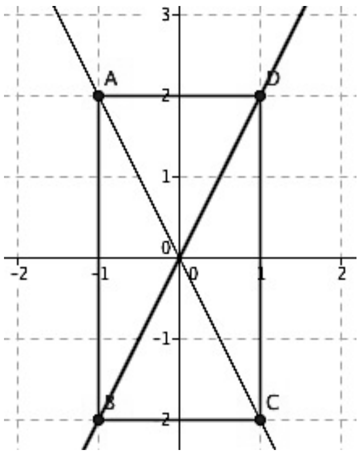
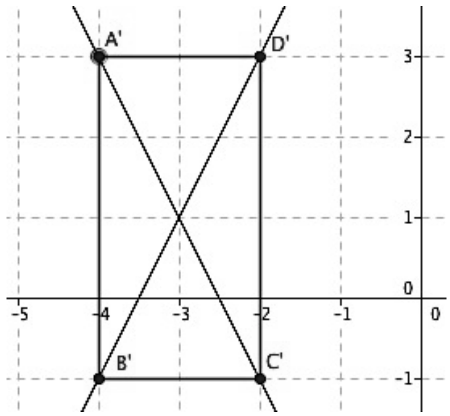


figure B



21. *figure* A

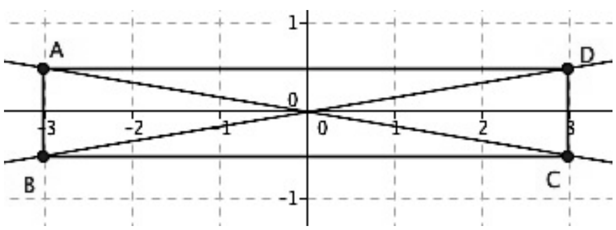
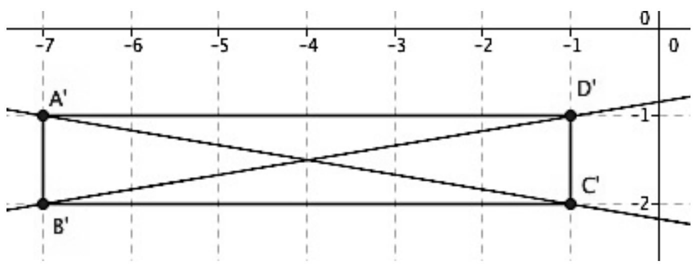


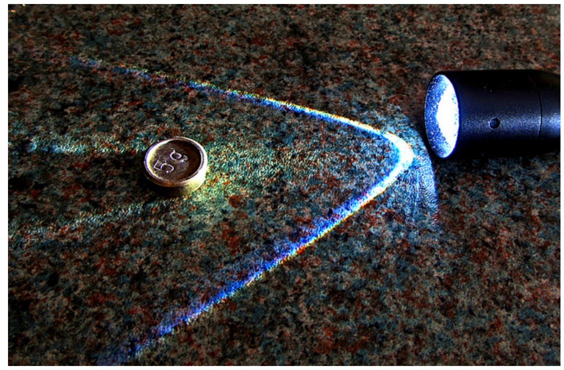
figure B



22. The equations of the diagonals of rectangle JKLM are $y_1 = 5/8x$ and $y_2 = -5/8x$. Rectangle JKLM is then translated so that its diagonals intersect at the point (12, -9). Write the equation of the diagonals of the translated rectangle.

6.11H What Happens If . . . ?

A Solidify Understanding Task



After spending some time working with circles and ellipses, Maya notices that the equations are a lot alike. For example, here's an equation of an ellipse and a circle:

$$\frac{x^2}{16}+\frac{y^2}{9}=1 \qquad x^2+y^2=16$$

1. What are some of the similarities between the circle and the ellipse given in the equations above? What are some of the differences?

2. Maya wonders what would happen if she took the equation of the circle and rearranged it so the right hand side was 1, like the standard form of an ellipse. What does the equation of the circle become?

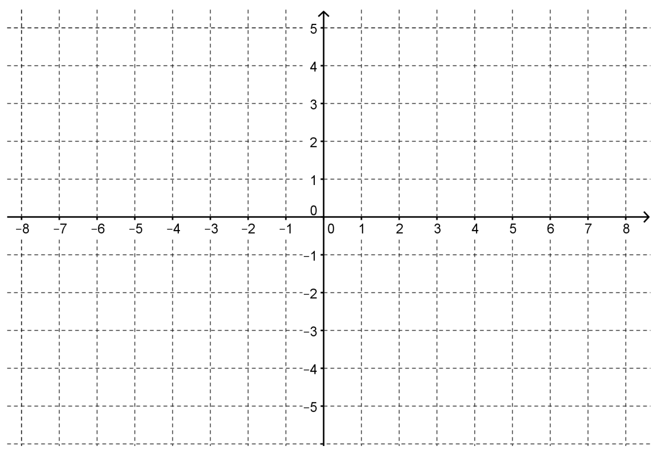
3. After seeing this equation Maya wonders if a circle is really an ellipse, or if an ellipse is really a circle. How would you answer this question?

4. Maya looks at the equation of the ellipse and wonders what would happen if the "+" in the equation was replaced with a "—", making the equation:

$$\frac{x^2}{16} - \frac{y^2}{9} = 1$$

Without making any further calculations or graphing any points, predict whether or not the graph of this equation will be an ellipse? Using what you know about ellipses, explain your answer.

5. Graph the equation to determine whether or not your prediction was correct. Be sure to use enough points to get a full picture of the figure.

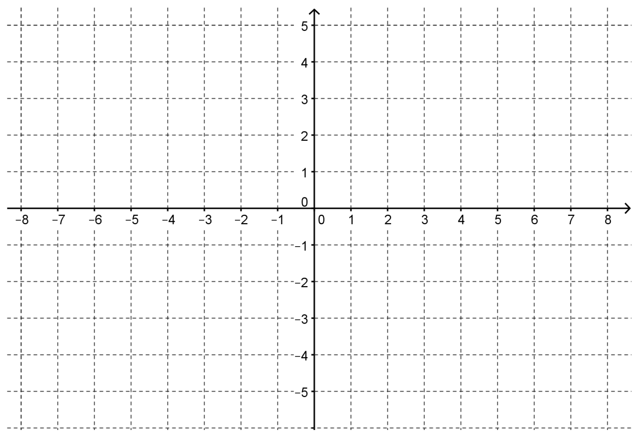


6. What are some of the features of the figure that you have graphed?

7. Maya's teacher tells her that the name of the figure represented in each of the two equations is a hyperbola. Maya wonders what would happen if the x^2 term in the equation was switched with the y^2 term, making the equation:

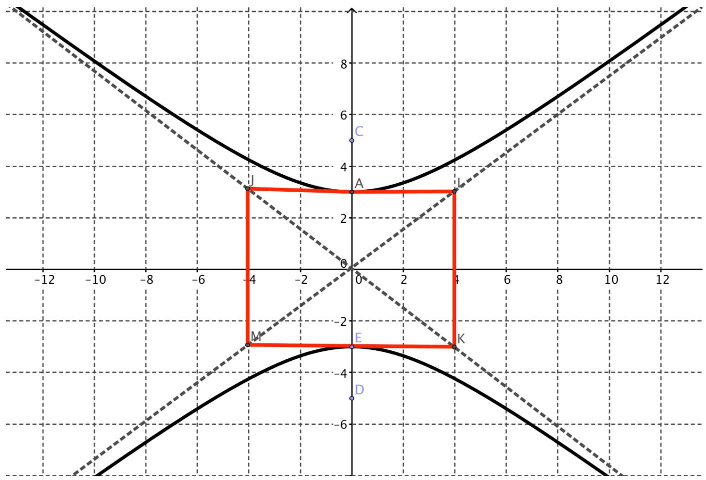
$$\frac{y^2}{9} - \frac{x^2}{16} = 1$$

Graph this equation and compare it to the hyperbola that you graphed previously.



8. What similarities and differences do you see between this hyperbola and the one that you graphed in #5?

One strategy that makes it easier to graph the hyperbola from an equation is to notice that the square root of the numbers under the x^2 and y^2 terms can be used to make a rectangle and then to draw dotted lines through the diagonals that form the boundaries of the hyperbola. Using this strategy to graph the equation: $\frac{y^2}{9} - \frac{x^2}{16} = 1,$ you would start by taking the square root of 9 = 3 and going up and down 3 units from the origin. Then you take the square root of 16 = 4 and go left and right 4 units from the origin. Make a rectangle with these points on the sides and draw the diagonals. You will get this:



9. So, Maya, the bold math adventurer, decides to try it with a new equation of a hyperbola. The standard form of the equation of an hyperbola centered at (0,0) is:

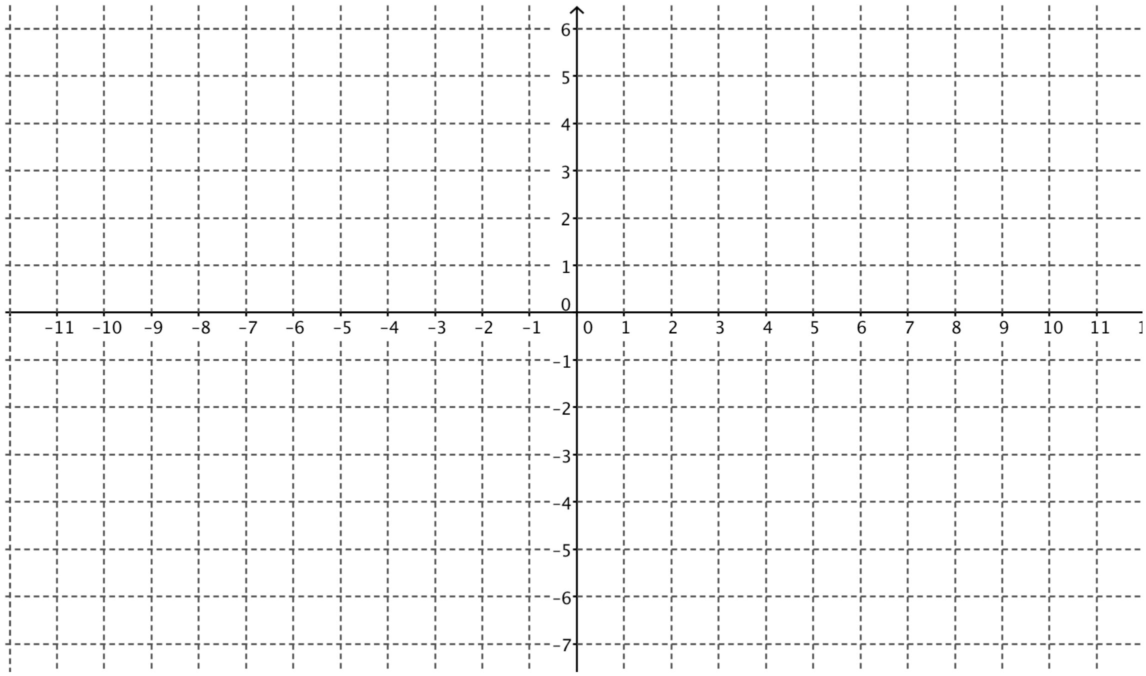
$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ (opens left and right)

$\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$ (opens up and down)

Maya goes to work graphing the equation:

$$\frac{x^2}{36} - \frac{y^2}{25} = 1$$

Try it yourself on the graph that follows and see what you can come up with.

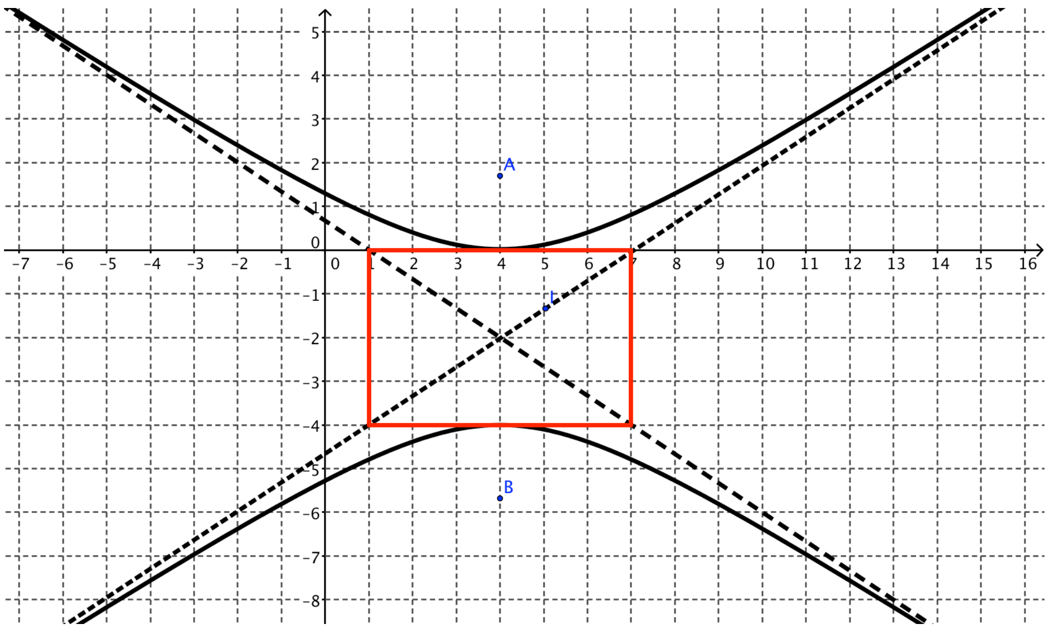


10. Maya wonders what happens if the equation becomes:

$$\frac{(x-1)^2}{36} - \frac{(y+2)^2}{25} = 1$$

What is your prediction? Why?

11. Write the equation of the hyperbola shown below:



12. What similarities and differences do you see between a hyperbola and an ellipse?

6.11H What Happens If ….?– Teacher Notes

A Solidify Understanding Task

Purpose:

The purpose of this task is to develop the definition of a hyperbola as the set of all points in a plane such that the difference between the distances from the point to each of the two foci is constant. The task is designed so that students draw upon their previous work with constructing ellipses and working with their equations to extend to equations of hyperbolas. The task begins with circles and ellipses and then asks them to consider what type of graph would be created if the equation was changed from addition to subtraction. The class discussion includes identifying the foci and the introducing the definition of the hyperbola.

Core Standards Focus:

G.GPE Expressing Geometric Properties With Equations

Translate Between The Geometric Description And The Equation For A Conic Section

G.GPE.3(+) Derive The Equations Of Ellipses And Hyperbolas Given The Foci, Using The Fact That The Sum Or Difference Of Distances From The Foci Is Constant.

Standards for Mathematical Practice:

SMP 3 – Construct viable arguments and critique the reasoning of others

SMP 8 – Look for express regularity in repeated reasoning

The Teaching Cycle:

Launch (Whole Class):

The task begins by asking students to consider their previous work with ellipses and circles, making comparisons between the two figures. Ask students to work questions 1-3 and then have a short discussion of their conclusions about the relationship between circles and ellipses. Remind students of the definition of the ellipse and ask students where the foci must be for a circle.

Ask students to read questions #4 and to make predictions about the features of the graph based on the equation. Will the graph be symmetrical about the x-axis? Will the graph be symmetrical about the y-axis? Will the graph be a function? After making predictions have students work problems 4- 6.

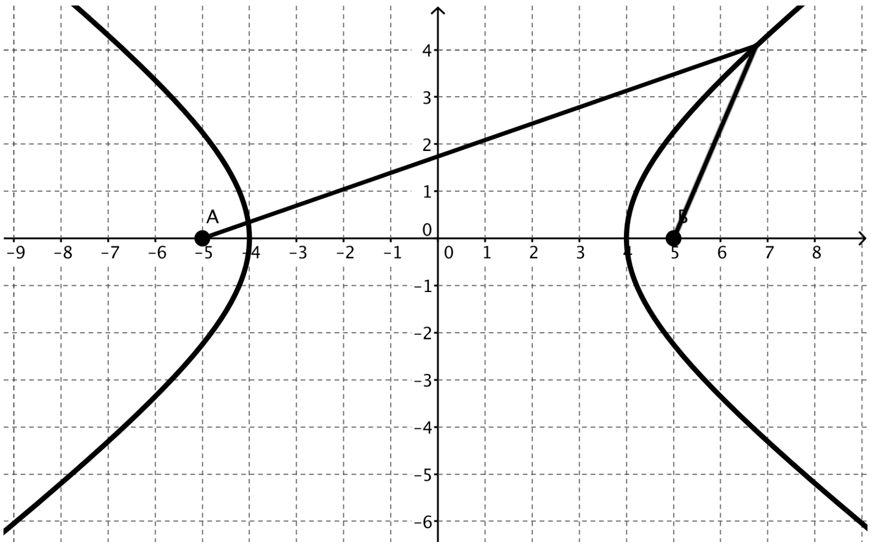
Explore (Small Group):

Monitor students as they work. You may need to support students in using the equation to plot points. The task will be very tedious if they are not using calculators to do the arithmetic necessary. You may choose to discourage the use of the graphing function of the calculator just so students think about how the symmetries occur using the equation. Some students will probably solve the equation for y (or x). Watch to see if they consider both the principal and the negative root of the expression. If they don't use the negative root, they will miss half of the graph, which would be an interesting discussion point when compared to students that used enough points to get the entire graph.

Discuss (Whole Class):

Ask a student to present their graph. You may wish to start with a group that only found part of the graph because they didn't do both the positive and negative root. When other students show the complete graph, discuss why the strategy of the solving for y may not yield the whole graph if the $\pm\sqrt{\ }$ isn't considered.

Explain to students that the figure that they have graphed is a hyperbola. Use technology to project the graph for the class and ask them to describe the features. Tell students that hyperbolas are also defined based upon the distances from a point to each of two foci. An ellipse is the set of all points such that the *sum* of the distances to the foci is constant. The hyperbola is the set of all points such that the *difference* of the distances to the foci is constant. The foci of this hyperbola are at (5,0) and (-5,0) and the segments showing the distances between the foci and a point of the hyperbola are shown below.



At this point, ask students to complete the task. Monitor their work and then call them back to discuss the remainder of the task. During the discussion of #9, help students clarify the use of a and b in the equation and using them to draw the rectangle that aids in graphing the hyperbola. Then, tell them that they can also be used to find the foci, with the relationship $a^2 + b^2 = c^2$, given that c is the distance from the center of the hyperbola to a focus. You may also wish to introduce the term, "asymptote" to describe the lines that are used to define the "borders" of the hyperbola.

Aligned Ready, Set, Go: Connecting Algebra & Geometry 6.11H

READY, SET, GO! Name Period Date

READY

Topic: Identifying Conic Sections by their equations

Identify each conic section by the given equation.

1. $\frac{x^2}{25}+\frac{y^2}{12}=1$

2. $\frac{x^2}{4}-\frac{y^2}{16}=1$

3. $\frac{x^2}{49}+\frac{y^2}{49}=1$

4. $x^2=16+y$

5. $9x^2=36+4y^2$

6. $9x^2=36-9y^2$

7. $y=\frac{x+4}{y}$

8. $7x^2-8y^2=35$

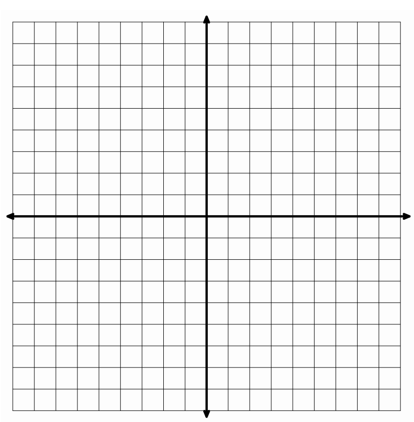
9. $5x^2-2y^2-15=-6y^2+5$

SET

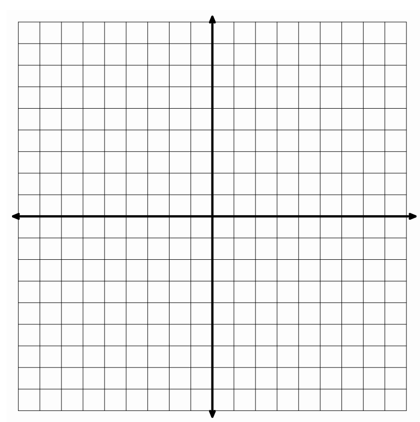
Topic: Graphing hyperbolas

Write the equation of the asymptotes. Then sketch the graph of the given equation.

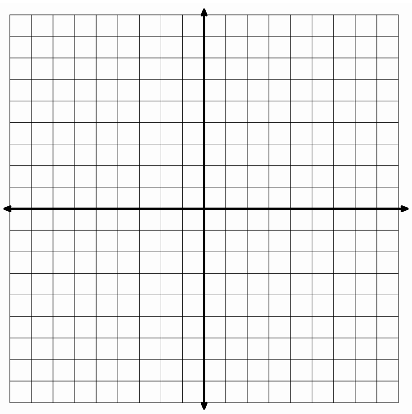
10. $\frac{x^2}{16}-\frac{y^2}{25}=1$



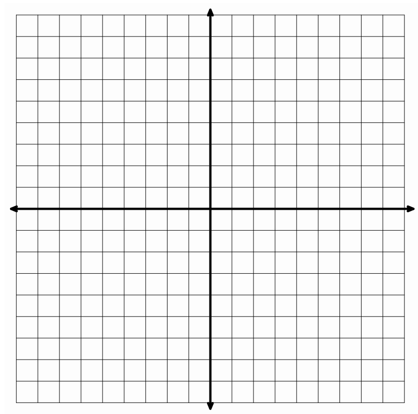
11. $\frac{y^2}{16}-\frac{x^2}{25}=1$



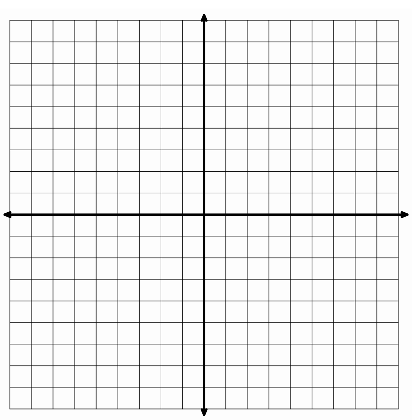
12. $\frac{y^2}{9} - \frac{x^2}{4} = 1$



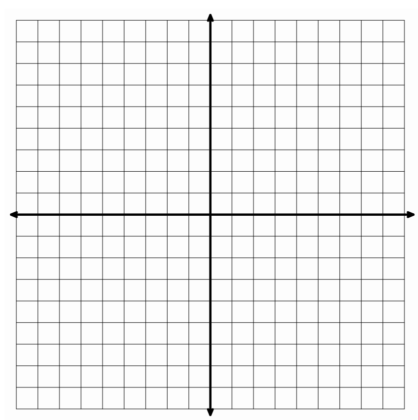
13. $\frac{x^2}{49} - \frac{y^2}{36} = 1$



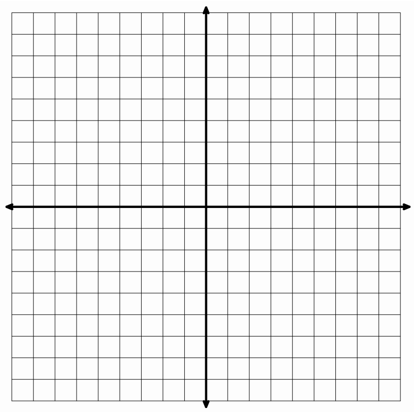
14. $4x^2 - 16y^2 = 64$



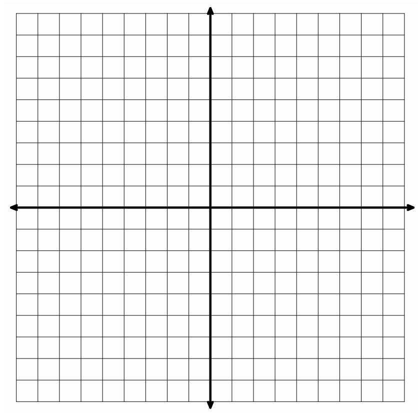
15. $12x^2 - 3y^2 = 48$



16. $\frac{(x-3)^2}{16} - \frac{(y+2)}{9} = 1$



17. $\frac{(y-5)^2}{4} - \frac{(x-3)}{9} = 1$



GO

Topic: Writing equations of conic sections in standard form

Write the equation in standard form by completing the square. Then identify the conic section. If the conic is:

- **a parabola, identify the vertex and the write the equation of the directrix.**
- **a circle, identify the center and the radius.**
- **an ellipse, identify the center and the radius for the horizontal and vertical axis.**
- **a hyperbola, write the equations of the asymptotes.**

18. $x^2 - 4x + y^2 + 6y = 1$

19. $16x^2 - 9y^2 - 72y - 288 = 0$

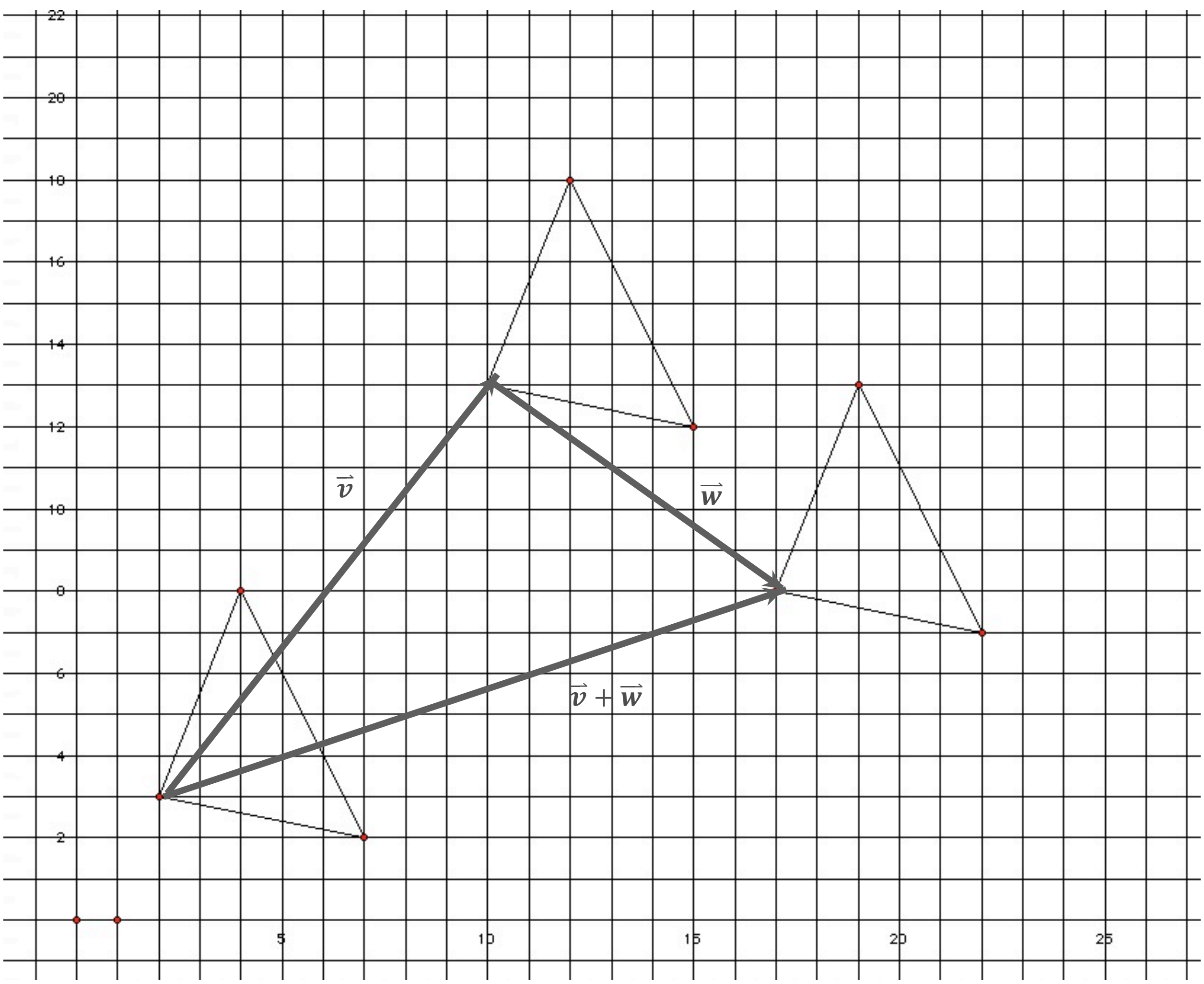
20. $2y^2 - 32x + 20y + 50 = 0$

21. $4x^2 + y^2 + 16x - 6y + 9 = 16$

6.12H The Arithmetic of Vectors

A Solidify Understanding Task

The following diagram shows a triangle that has been translated to a new location, and then translated again. Arrows have been used to indicate the movement of one of the vertex points through each translation. The result of the two translations can also be thought of as a single translation, as shown by the third arrow in the diagram.



Draw arrows to show the movement of the other two vertices through the sequence of translations, and then draw an arrow to represent the resultant single translation. What do you notice about each set of arrows?

A **vector** is a quantity that has both **magnitude** and **direction**. The arrows we drew on the diagram represent translations as vectors—each translation has *magnitude* (the distance moved) and *direction* (the direction in which the object is moved). Arrows, or *directed line segments*, are one way of representing a vector.

Addition of Vectors

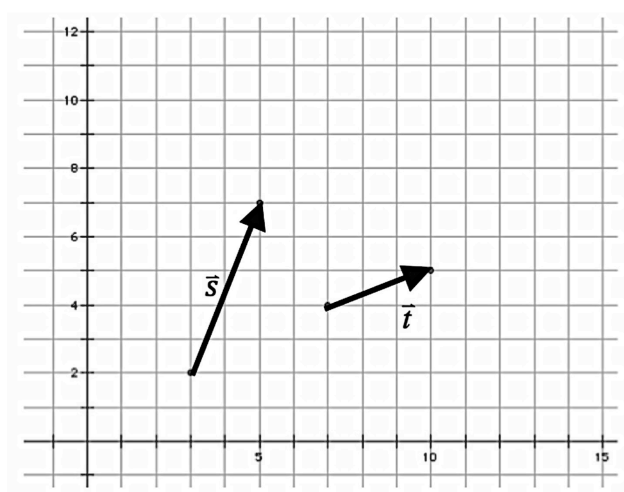
In the example above, two vectors $\vec{v}$ and $\vec{w}$ were combined to form vector $\vec{r}$. This is what is meant by "adding vectors".

1. Study each of the following three methods for adding vectors, then try each method to add vectors $\vec{s}$ and $\vec{t}$ given in the diagrams to find $\vec{q}$, such that $\vec{s} + \vec{t} = \vec{q}$.

2. Explain why each of these methods gives the same result.

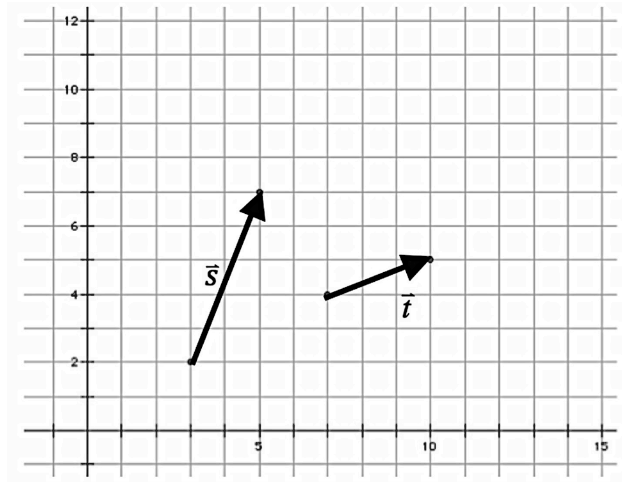
Method 1: *End-to-end*

The diagram given above illustrates the end-to-end strategy of adding two vectors to get a resultant vector that represents the sum of the two vectors. In this case, the resulting vector shows that a single translation could accomplish the same movement as the combined sum of the two individual translations, that is $\vec{v} + \vec{w} = \vec{r}$.



Method 2: *The parallelogram rule*

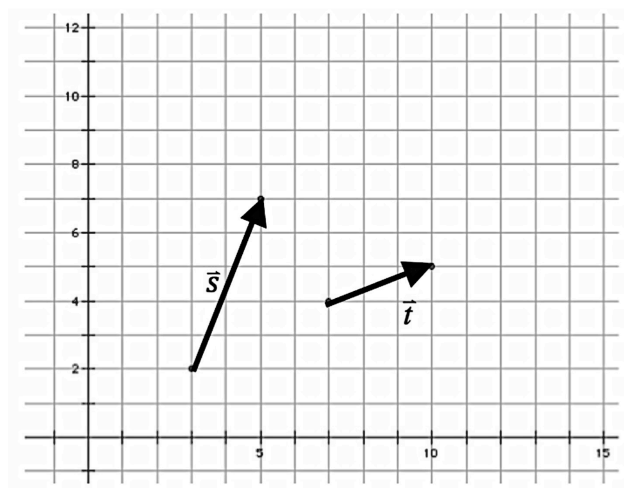
Since we can relocate the arrow representing a vector, draw both vectors starting at a common point. Often both vectors are relocated so they have their *tail* ends at the origin. These arrows form two sides of a parallelogram. Draw the other two sides. The resulting sum is the vector represented by the arrow drawn from the common starting point (for example, the origin) to the opposite vertex of the parallelogram.



Question to think about: How can you determine where to put the missing vertex point of the parallelogram?

Method 3: *Using horizontal and vertical components*

Each vector consists of a horizontal component and a vertical component. For example, vector $\vec{v}$ can be thought of as a movement of 8 units horizontally and 13 units vertically. This is represented with the notation $\langle 8,13\rangle$. Vector $\vec{w}$ consists of a movement of 7 units horizontally and -5 units vertically, represented by the notation $\langle 7,-5\rangle$.



Question to think about: How can the components of the individual vectors be combined to determine the horizontal and vertical components of the resulting vector $\vec{r}$?

3. Examine vector $\vec{s}$ given above. While we can relocate the vector, in the diagram the *tail* of the vector is located at (3, 2) and the *head* of the vector is located at (5, 7). Explain how you can determine the horizontal and vertical components of a vector from just the coordinates

of the point at the tail and the point at the head of the vector? That is, how can we find the horizontal and vertical components of movement without counting across and up the grid?

Magnitude of Vectors

The symbol $\|\vec{v}\|$ is used to denote the magnitude of the vector, in this case the length of the vector.

Devise a method for finding the magnitude of a vector and use your method to find the following.

Be prepared to describe your method for finding the magnitude of a vector.

4. $\|\vec{v}\|$

5. $\|\vec{w}\|$

6. $\|\vec{v} + \vec{w}\|$

Scalar Multiples of Vectors

We can stretch a vector by multiplying the vector by a scale factor. For example, $2\vec{v}$ represents the vector that has the same direction as $\vec{v}$, but whose magnitude is twice that of $\vec{v}$.

Draw each of the following vectors on a coordinate graph:

7. $3\vec{s}$

8. $-2\vec{t}$

9. $3\vec{s} + (-2\vec{t})$

10. $3\vec{s} - 2\vec{t}$

Other Applications of Vectors

We have illustrated the concept of a vector using translation vectors in which the magnitude of the vector represents the distance a point gets translated. There are other quantities that have magnitude and direction, but the magnitude of the vector does not always represent length.
For example, a car traveling 55 miles per hour along a straight stretch of highway can be represented by a vector since the speed of the car has magnitude, 55 miles per hour, and the car is traveling in a specific direction. Pushing on an object with 25 pounds of force is another example. A vector can be used to represent this push since the force of the push has magnitude, 25 pounds of force, and the push would be in a specific direction.

11. A swimmer is swimming across a river with a speed of 20 ft/sec and at a 45° angle from the bank of the river. The river is flowing at a speed of 5 ft/sec. Illustrate this situation with a vector diagram and describe the meaning of the vector that represents the sum of the two vectors that represent the motion of the swimmer and the flow of the river.

12. Two teams are participating in a tug-of-war. One team exerts a combined force of 200 pounds in one direction while the other team exerts a combined force of 150 pounds in the other direction. Illustrate this situation with a vector diagram and describe the meaning of the vector that represents the sum of the vectors that represent the efforts of the two teams.

6.12H The Arithmetic of Vectors – Teacher Notes

A Solidify Understanding Task

Teacher Note: This task contains many conceptual ideas, strategic procedures, and representational ways of thinking about vectors, and you may find it to be too much to attempt in one class period. If you choose to separate the task into two parts, the first day should consist of the three methods for adding vectors, a discussion of why the three methods produce the same results (question 2), and end with a discussion of how to find the horizontal and vertical components of a vector (question 3). The second day would include strategies for finding the magnitude of a vector, drawing scalar multiples of vectors and defining subtraction as adding the vector that faces the opposite direction. The second day would also examine vector quantities such as velocity or force, situations in which the length of the directed line segment used to represent the vector quantity measures something other than distance.

Purpose: Students already have an intuitive understanding of one application of vectors—the translation vector—based on their work with translations of figures in a plane. The purpose of this task is to make the concept of a translation vector explicit (i.e., a translation vector has both magnitude and direction), and then to use translation vectors to examine some of the arithmetic of vectors: adding and subtracting vectors, and scalar multiplication of vectors. Three methods for adding vectors are introduced: end-to-end, the parallelogram rule, and using horizontal and vertical components of the vector. The last part of the task considers other possible applications of vectors—quantities that have both magnitude and direction—such as velocity and force.

Core Standards Focus:

N.VM.1 Recognize vector quantities as having both magnitude and direction. Represent vector quantities by directed line segments, and use appropriate symbols for vectors and their magnitudes (e.g., $\mathbf{v}$, $|\mathbf{v}|$, $||\mathbf{v}||$, v).

N.VM.2 Find the components of a vector by subtracting the coordinates of an initial point from the coordinates of a terminal point.

N.VM.3 Solve problems involving velocity and other quantities that can be represented by vectors.

N.VM.4 Add and subtract vectors.

a. Add vectors end-to-end, component-wise, and by the parallelogram rule. Understand that the magnitude of a sum of two vectors is typically not the sum of the magnitudes.
b. Given two vectors in magnitude and direction form, determine the magnitude and direction of their sum.
c. Understand vector subtraction **v** – **w** as **v** + (–**w**), where –**w** is the additive inverse of **w**, with the same magnitude as **w** and pointing in the opposite direction. Represent vector subtraction graphically by connecting the tips in the appropriate order, and perform vector subtraction component-wise.

N.VM.5 Multiply a vector by a scalar.

a. Represent scalar multiplication graphically by scaling vectors and possibly reversing their direction; perform scalar multiplication component-wise, e.g., as $c(v_x, v_y) = (cv_x, cv_y)$.
b. Compute the magnitude of a scalar multiple c**v** using ||c**v**|| = |c|**v**. Compute the direction of c**v** knowing that when |c|**v** ≠ 0, the direction of c**v** is either along **v** (for $c > 0$) or against **v** (for $c < 0$).

Standards for Mathematical Practice:

SMP 7 – Look for and make use of structure

SMP 4 – Model with mathematics

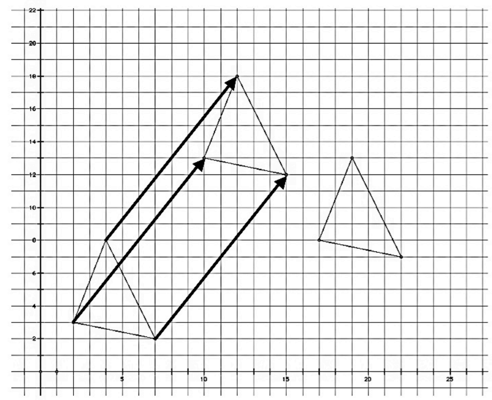
The Teaching Cycle:

Launch (Whole Class): [questions 1-3]

Discuss the diagram on the first page of the task, including the additional arrows students are asked to draw in question 1. The discussion should bring out the following points:

- A **vector** is a quantity that has both magnitude and direction. The arrows we drew on the diagram represent the translations as vectors—each translation has *magnitude* (the distance moved) and *direction* (the direction in which the object is moved).
- Arrows, or *directed line segments*, are one way of representing a vector.
- For each translation, the arrows connecting a pre-image point to its image point are the same length and parallel to each other.

- We can draw an infinite set of such arrows for each translation since each point in the plane—including points in the interior and exterior of the triangle, as well as points on each of the line segments forming the triangle—get translated the same distance and same direction.
- While the vector can be represented by many different parallel line segments of the same length, there is only one vector. That is, a vector has magnitude and direction, but no location. In the diagram, each arrow drawn represents the same vector, even though the arrows are drawn in three separate locations.



Before moving to the explore, point out ways of naming vectors using the "harpoon" over the top of the variable, or using boldfaced text. Also point out the component form of representing a vector used in addition method 3 of the task. Assign students to work on the first part of the task, through question 3.

Explore (Small Group): [questions 1-3]

After clarifying that a vector is a quantity that has both magnitude and direction, and that vectors can be represented by directed line segments (arrows) drawn anywhere in the plane, turn students attention to the strategies for adding vectors as described in the task. Allow students time to try out each strategy and select students to present who can articulate how they applied each strategy to find $\vec{s} + \vec{t}$. This work should lead students to the observation that in each of these methods the horizontal components of movement of the individual vectors are combined together and the vertical components are also combined. Along with the third method for adding vectors using components, students should consider question 3: How can we find these components without counting the horizontal and vertical movement across and up the grid?

Discuss (Whole Class): [questions 1-3]

Discuss each of the strategies for adding vectors by having selected students present their work. Discuss the questions at the end of the sections describing the parallelogram rule and the component-wise methods for adding vectors. These questions should point students' attention towards adding the horizontal components together and then the vertical components together from each of the two addend vectors in order to find the horizontal and vertical components of the resultant vector. The presentations should include a summary discussion as to why these three methods produce the same results.

Question 3 should suggest a method for writing a vector in component form. Students may observe that if the coordinates of the point at the tail of the vector are (x_1, y_1) and the coordinates of the point at the head of the vector are (x_2, y_2), then the horizontal component of the vector is given by $x_2 - x_1$ and the vertical component is given by $y_2 - y_1$. Therefore, the component form of a vector is given by $\langle x_2 - x_1, y_2 - y_1\rangle$.

Launch (Whole Class): [questions 4-11]
Following the presentations and discussions about adding vectors, assign students to work on the remainder of the task with a partner.

Explore (Small Group): [questions 4-11]
If students are having difficulty developing a strategy for finding the magnitude (i.e., length) of a vector in questions 4-6, remind them of previous work in which they have treated a non-vertical, non-horizontal line segment as the hypotenuse of a right triangle so they could apply the Pythagorean Theorem to find its length. They might also draw upon recent work with the distance formula.

The work with scalar multiples in questions 7-10 will surface the idea of changing the direction of a vector when multiplying by a negative scalar factor, and defining subtraction as adding the vector facing the opposite direction. Watch for students who are surfacing these ideas in their discussions with their peers.

Students who have successfully added vectors using all three methods and have worked out a strategy for finding the magnitude of a vector and for drawing scalar multiples of vectors can work on the additional applications of vectors in questions 11 and 12. These applications give students a sense of how vectors can represent quantities that have magnitude and direction, but for which the magnitude of the vector represents something other than distance, such as speed or force.

Discuss (Whole Class): [questions 4-11]

Have students present their strategies for finding the magnitude or length of a vector, which should be based on the Pythagorean theorem or the distance formula. Also, discuss scalar multiples of a vector, including the reversal of direction when multiplying by a negative scalar. End this discussion of scalars by defining subtraction of vectors as adding the additive inverse (i.e., the vector with the same magnitude but opposite direction).

If there is time, examine the additional application problems as a whole class. It is important that students scale their vectors so that the length of the arrow used to represent each quantity is proportionally correct. For example, in the swimming problem the vector representing the speed of the swimmer should be 4 times longer than the vector representing the speed of the water and drawn at a 45° angle to the vector representing the downstream flow of the water.

Aligned Ready, Set, Go: Connecting Algebra and Geometry 6.12H

READY, SET, GO! Name Period Date

READY

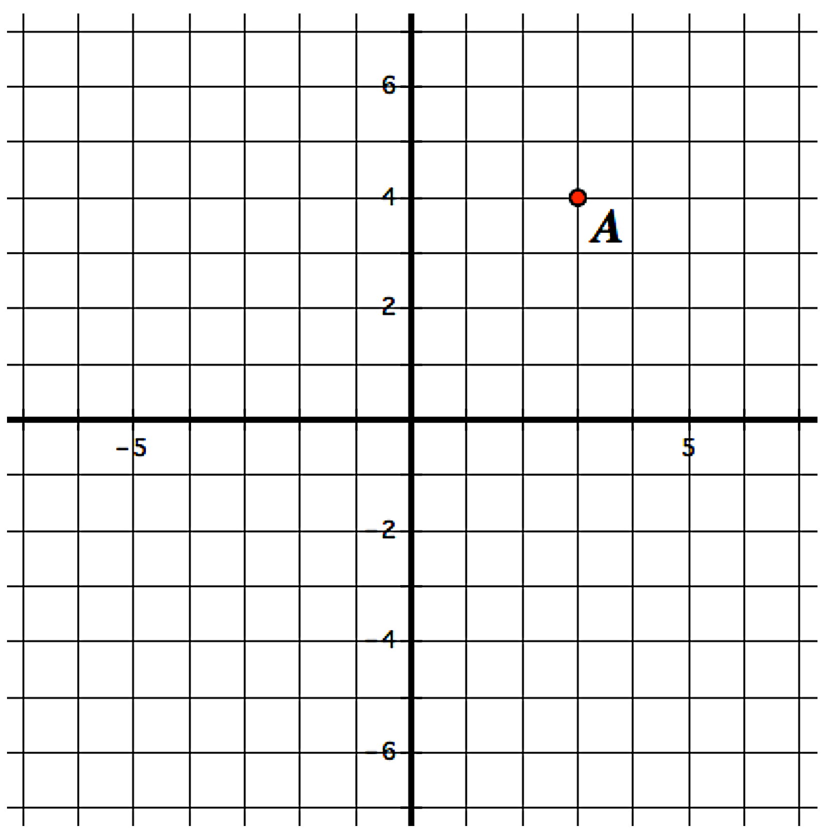
Topic: Rotations on a coordinate grid

Given Point *A* (3, 4). Plot the points described below on the coordinate grid:

1. Point *B* is the image of point *A* after a 90° rotation counterclockwise about the origin

2. Point *C* is the image of point *A* after a 180° rotation counterclockwise about the origin

3. Write the equation of the circle that contains points *A, B* and *C.*



SET

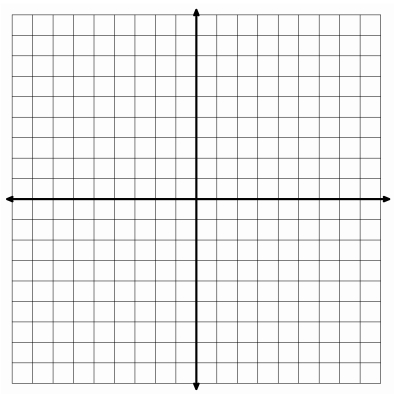
Topic: Adding vectors

Two vectors are described in component form in the following way:

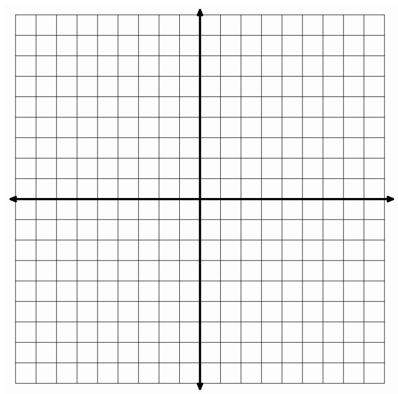
$$\vec{v} : \langle -2, 3 \rangle \text{ and } \vec{w} : \langle 3, 4 \rangle$$

On the grids below, create vector diagrams to show:

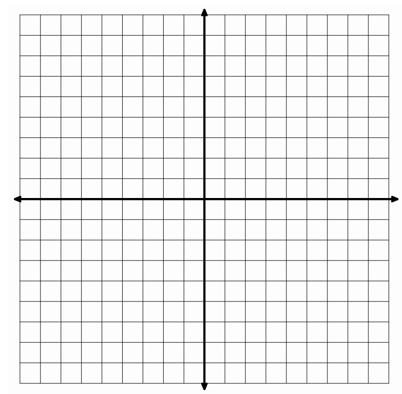
4. $\vec{v} + \vec{w} =$



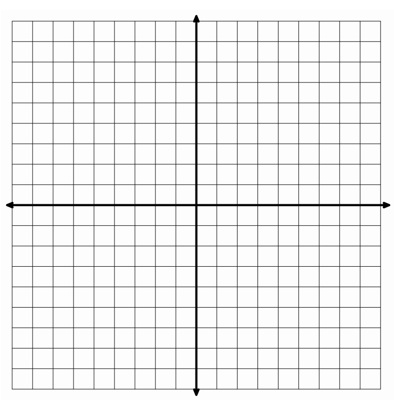
5. $\vec{v} - \vec{w} =$



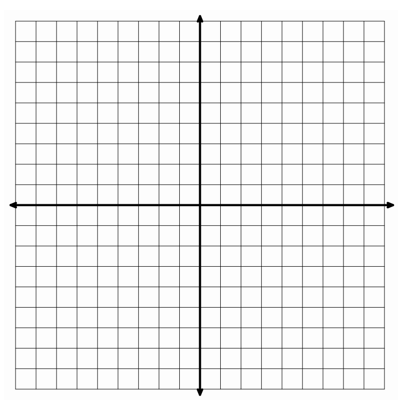
6. $3\vec{v} =$



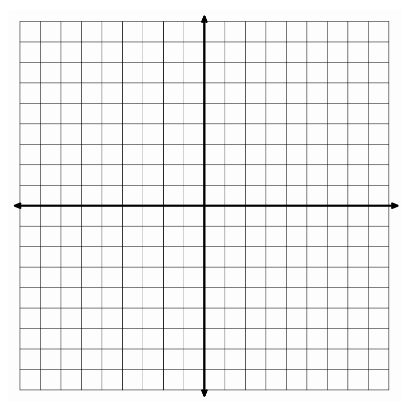
7. $-2\vec{w} =$



8. $3\vec{v} - 2\vec{w} =$



9. Show how to find $\vec{v} + \vec{w}$ using the parallelogram rule.



GO

Topic: The arithmetic of matrices

$$A = \begin{bmatrix} 2 & -3 \\ -1 & 5 \end{bmatrix},\ B = \begin{bmatrix} 2 & 5 \\ -3 & 2 \end{bmatrix},\ \textbf{and } C = \begin{bmatrix} 4 & 2 & -1 \\ 5 & 2 & 3 \end{bmatrix}$$

Find the following sums, differences or products, as indicated. If the sum, difference or product is undefined, explain why.

10. $A + B$

11. $A + C$

12. $2A - B$

13. $A \cdot B$

14. $B \cdot A$

15. $A \cdot C$

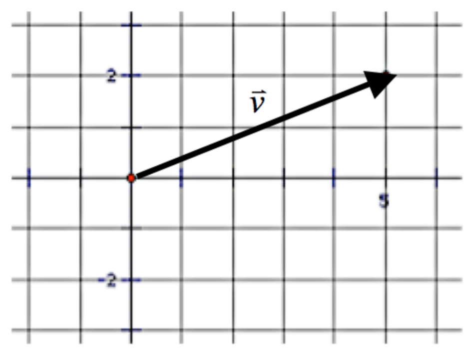
16. $C \cdot A$

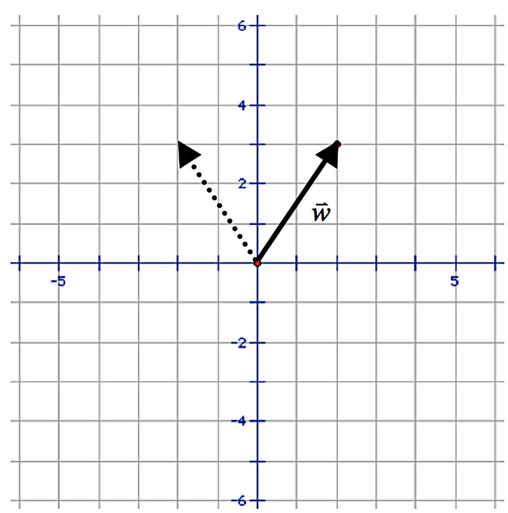
6.13H Transformations with Matrices

A Solidify Understanding Task

Various notations are used to denote vectors: bold-faced type, ***v***; a variable written with a harpoon over it, $\vec{v}$; or listing the horizontal and vertical components of the vector, $\langle v_x, v_y \rangle$. In this task we will represent vectors by listing their horizontal and vertical components in a matrix with a single column, $\begin{bmatrix} v_x \\ v_y \end{bmatrix}$.

1. Represent the vector labeled $\vec{v}$ in the diagram at the right as a matrix with one column.





Matrix multiplication can be used to transform vectors and images in a plane.

Suppose we want to reflect $\vec{w}$ over the y-axis. We can represent $\vec{w}$ with the matrix $\begin{bmatrix} 2 \\ 3 \end{bmatrix}$, and the reflected vector with the matrix $\begin{bmatrix} -2 \\ 3 \end{bmatrix}$.

2. Find the 2×2 matrix that we can multiply the matrix representing the original vector by in order to obtain the matrix that represents the reflected vector. That is, find *a, b, c* and *d* such that $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} -2 \\ 3 \end{bmatrix}$.

3. Find the matrix that will reflect $\vec{w}$ over the *x*-axis.

4. Find the matrix that will rotate $\vec{w}$ 90° counterclockwise about the origin.

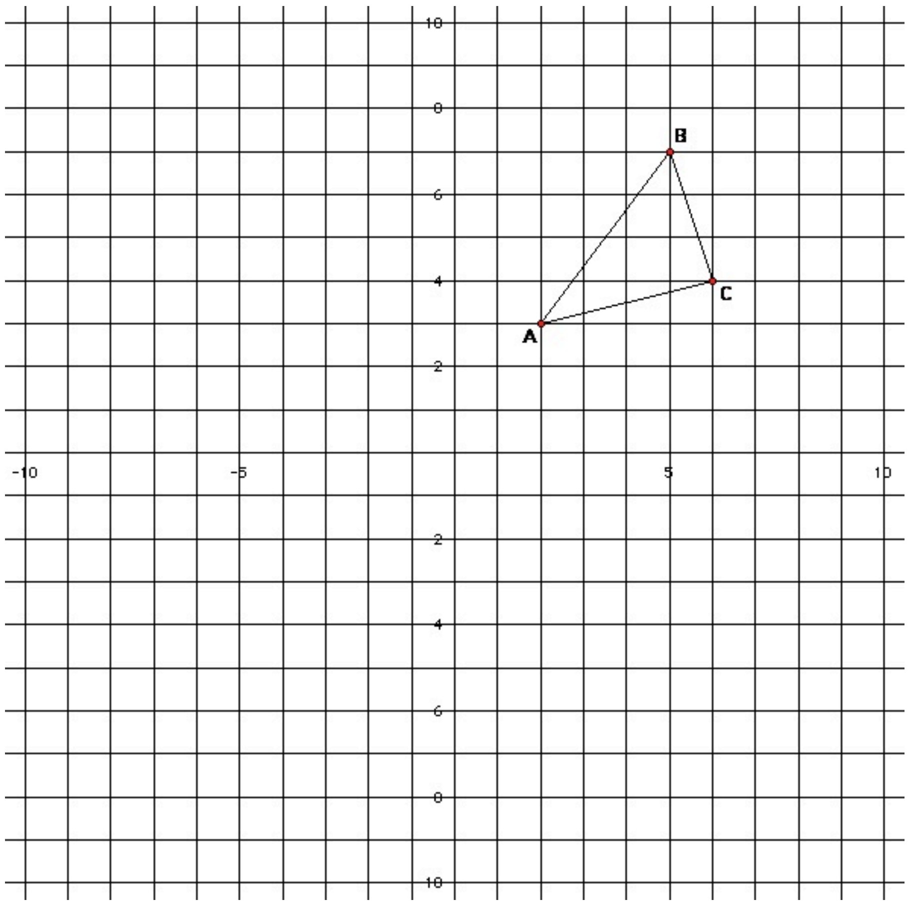
5. Find the matrix that will rotate $\vec{w}$ 180° counterclockwise about the origin.

6. Find the matrix that will rotate $\vec{w}$ 270° counterclockwise about the origin.

7. Is there another way to obtain a rotation of 270° counterclockwise about the origin other than using the matrix found in question 6? If so, how?

We can represent polygons in the plane by listing the coordinates of its vertices as columns of a matrix.

For example, the triangle below can be represented by the matrix $\begin{bmatrix} 2 & 5 & 6 \\ 3 & 7 & 4 \end{bmatrix}$.



8. Multiply this matrix, which represents the vertices of ΔABC, by the matrix found in question 2. Interpret the product matrix as representing the coordinates of the vertices of another triangle in the plane. Plot these points and sketch the triangle. How is this new triangle related to the original triangle?

9. How might you find the coordinates of the triangle that is formed after ΔABC is rotated 90° counterclockwise about the origin using matrix multiplication? Find the coordinates of the rotated triangle.

10. How might you find the coordinates of the triangle that is formed after ΔABC is reflected over the x-axis using matrix multiplication? Find the coordinates of the reflected triangle.

6.13H Transformations with Matrices – Teacher Notes

A Solidify Understanding Task

Purpose: In this task students will examine the use of matrices to produce transformations. Vectors will be represented as matrices with a single column. Students will determine matrices that transform a vector in one of the following ways when the vector matrix is multiplied by the appropriate transformation matrix.

- the vector is reflected over the y-axis
- the vector is reflected over the x-axis
- the vector is rotated 90° about the origin
- the vector is rotated 180° about the origin

Students will use these same matrices to transform polygons by multiplying a matrix that contains the coordinates of the vertices of the original polygon by the appropriate transformation matrix.

Core Standards Focus:

N.VM.11 Multiply a vector (regarded as a matrix with one column) by a matrix of suitable dimensions to produce another vector. Work with matrices as transformations of vectors.

N.VM.12 Work with 2×2 matrices as transformations of the plane.

Related Standards: G.CO.2, G.CO.5

Standards for Mathematical Practice:

SMP 7 – Look for and make use of structure

The Teaching Cycle:

Launch (Whole Class): [questions 3-7]

Review with your students the information about notation for representing vectors, as given on the first page of the task. Make sure students understand how a vector can be represented as a matrix with one column. You may want to review the row-by-column procedure for matrix multiplication that was developed in a previous module. Also remind students of the definitions of rotations and reflections and point out that in this task the *x*- and *y*-axes will serve as lines of reflection, and that rotations will be about the origin.

Once students are comfortable with the matrix representation of a vector, and have reviewed the appropriate skills and concepts used in this task, present the issue of finding a matrix that can be used as a factor along with the matrix representing a vector, so that the resulting product matrix represents the vector after it has been reflected about the *y*-axis (question 2). Give students a few minutes to work individually to find this matrix, and then share their work—both their final result as well as their strategy for finding this matrix (see potential strategies in the explore). Once strategies have been presented for finding the transformation matrix required in question 2, set students about the work of finding the other transformation matrices required on questions 3-6.

Explore (Small Group): [questions 3-7]

Two strategies that potentially will emerge for finding the transformation matrices include a guess-and-check strategy and a strategy built on writing and solving equations by inspection. For example, on question 2 students might use trial and error to find values for *a, b, c* and *d*—using such reasoning as, "I want to multiply 2 by -1 to change its sign, but then I don't want to add anything to it, so the other partial product needs to be 0." Or, students may be more systematic and multiply the two matrices on the left of the equal sign together and set the resulting expressions equal to the elements in the product matrix on the right. This will lead them to the following two equations, which can be solved by inspection:

$$2a + 3b = -2 \qquad 2c + 3d = 3$$

Students who finish questions 3-6 before other students can work on question 7.

Discuss (Whole Class): [questions 3-7]

Make a list of the matrices students found for each of the required transformations, as follows:

Desired transformation	Transformation matrix
Reflect over y-axis	$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$
Reflect over x-axis	$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$
Rotate 90° counterclockwise about the origin	$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$
Rotate 180° counterclockwise about the origin	$\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$
Rotate 270° counterclockwise about the origin	$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$

Discuss question 7. This question is intended to help students recognize that they can multiply two transformation matrices together to get a new matrix that represents the combined effect of the two individual transformations. For example, multiplying the matrix that rotates a vector 90° counterclockwise about the origin by the matrix that rotates a vector 180° counterclockwise about the origin produces the matrix that rotates a vector 270° counterclockwise about the origin. Likewise, using the matrix that rotates a vector 90° counterclockwise about the origin as a factor three times also produces the matrix that rotates a vector 270° counterclockwise about the origin, and using that matrix as a factor four times produces the identity matrix, which would return the vector to its original position after a rotation of 360°. Multiplying the matrix that reflects a vector over the *x*-axis by the matrix that reflects a vector over the y-axis produces the matrix that rotates a vector 180° counterclockwise about the origin.

Launch (Whole Class): [questions 8-10]
Use the given diagram on the third page of the task to show how vertices of a polygon can be represented by the columns of a matrix. Then set students to work on questions 8-10.

Explore (Small Group): [questions 8-10]
As students work on questions 8-10 they should observe that the same matrices we listed previously can be used to reflect or rotate a collection of points that represent the vertices of a polygon.

Discuss (Whole Class): [questions 8-10]
Ask students to describe how a single transformation matrix can be used to reflect or rotate a complete set of points representing the pre-image of the transformation.

Aligned Ready, Set, Go: Connecting Algebra and Geometry 6.13H

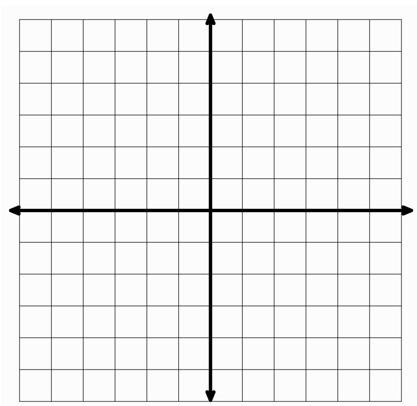
READY, SET, GO! Name Period Date

READY

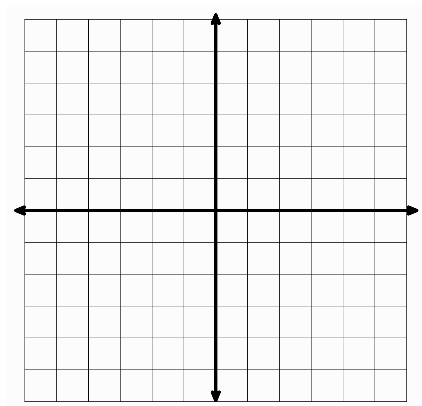
Topic: Adding Vectors

Given vectors $\vec{v}:\langle -2,\ 4\rangle$ **and** $\vec{w}:\langle 5,-2\rangle$**, find the following using the parallelogram rule:**

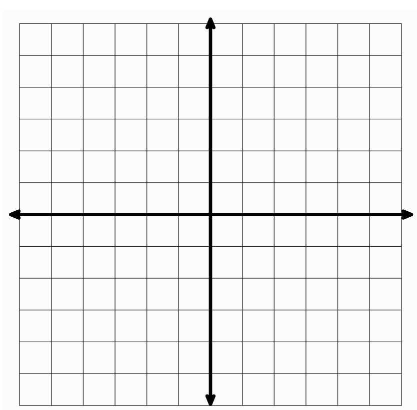
1. $\vec{v}+\vec{w}=$



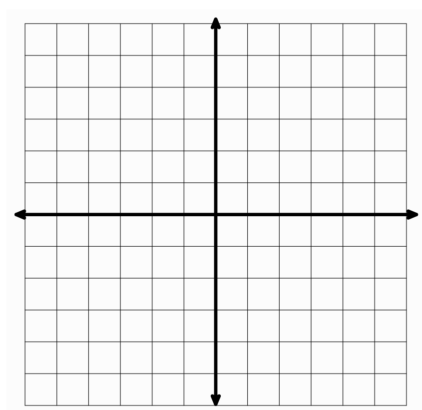
2. $\vec{v}-\vec{w}=$



3. $2\vec{v}+\vec{w}=$



4. $\vec{v}-2\vec{w}=$



SET

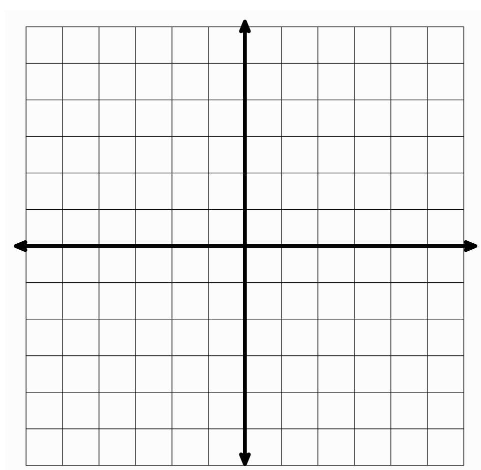
Topic: Matrices and Transformations of the Plane

5. List the coordinates of the four vertices of the parallelogram you drew in question 1 as a matrix. The x-values will be in the left column, and the y-values will be in the right column.

$$\begin{array}{cc} & \begin{matrix} \mathrm{x} & \quad & \mathrm{y} \end{matrix} \\ \begin{matrix}\text{Point 1}\\ \text{Point 2}\\ \text{Point 3}\\ \text{Point 4}\end{matrix} & \begin{bmatrix} \quad & \quad \\ & \\ & \\ & \end{bmatrix} \end{array}$$

6. Multiply the matrix you wrote in question 3 by the following matrix: $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

7. Plot the original parallelogram. Then, use the points from the matrix in number 6 and plot them on the following coordinate grid. Connect those 4 points. What transformation occurred between your original parallelogram and the new one?



GO

Topic: Transformation of Functions

Function $f(x)$ is defined by the following table below:

x	2	4	6	8	10	12	14	16
$f(x)$	-8	-3	2	7	12	17	22	27
$g(x)$								
$h(x)$								

8. Write an equation for $f(x)$.

9a. Fill in the values for $g(x)$ assuming that $g(x) = f(x) + 3$

b. Write an equation for $g(x)$.

10a. Fill in the values for $h(x)$ assuming that $h(x) = 2f(x)$

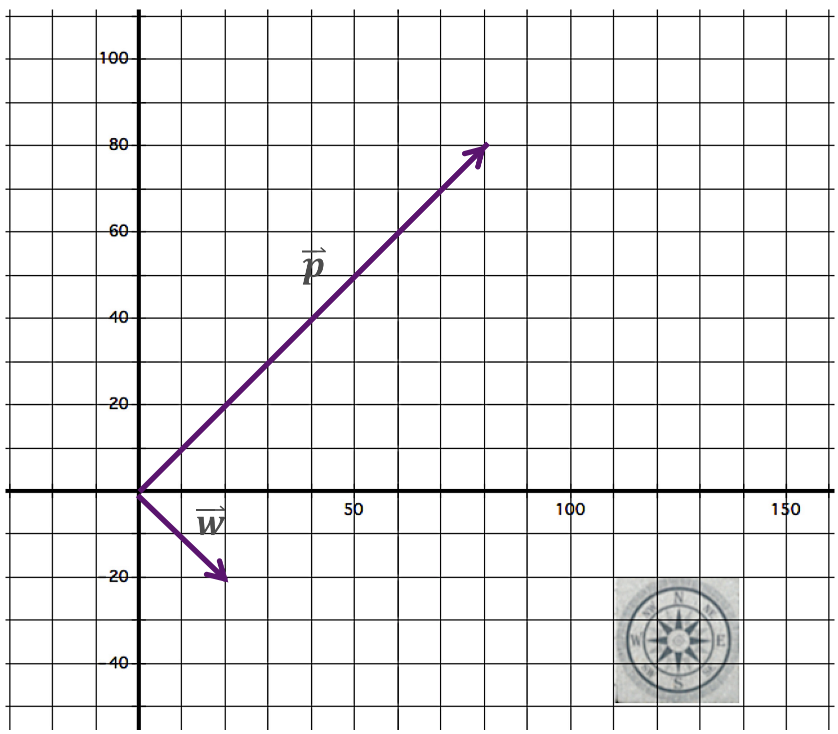
b. Write an equation for $h(x)$.

6.14H Plane Geometry

A Practice Understanding Task

Jon's father is a pilot and he is using vector diagrams to explain some principles of flight to Jon. His father has drawn the following diagram to represent a plane that is being blown off course by a strong wind. The plane is heading northeast as represented by $\vec{p}$ and the wind is blowing towards the southeast as represented by $\vec{w}$.

1. Based on this diagram, what is the plane's speed and what is the wind's speed? (The vector diagram represents the speed of the plane in still air.) Note: Each grid unit is 10 $^{miles}/_{hour}$



2. Use this diagram to find the ground speed of the plane, which will result from a combination of the plane's speed and the wind's speed. Also, indicate on the diagram the direction of motion of the plane relative to the ground.

3. Jon drew a parallelogram to determine the ground speed and direction of the plane. If you have not already done so, draw Jon's parallelogram and explain how it represents the original problem situation as well as the answers to the question asked in problem 2.

4. Write a matrix equation that will reflect the parallelogram you drew in problem 3 over the y-axis. Use the solution to the matrix equation to draw the resulting parallelogram.

5. Prove that the resultant figure of the reflection performed in problem 4 is a parallelogram. That is, explain how you know opposite sides of the resulting quadrilateral are parallel.

6. Find the area of the parallelogram drawn in problem 3. Explain your method for determining the area.

6.14H Plane Geometry– Teacher Notes

A Practice Understanding Task

Purpose: The purpose of this task is to practice using matrices to represent vectors, and matrix operations to represent transformations of the plane. In addition, students will practice using the distance formula to determine the magnitude of a vector, and interpret the meaning of the magnitude and orientation of a vector in terms of a real-world context. The properties of a parallelogram will be exploited through the use of the parallelogram rule for adding vectors, through finding the area of a parallelogram using the determinant of a 2×2 matrix, and by observing that parallelism is preserved under rigid-motion transformations.

Core Standards Focus:

N.VM.3 Solve problems involving velocity and other quantities that can be represented by vectors.

N.VM.4a Add vectors end-to-end, component-wise, and by the parallelogram rule. Understand that the magnitude of a sum of two vectors is typically not the sum of the magnitudes.

N.VM.12 Work with 2×2 matrices as transformations of the plane, and interpret the absolute value of the determinant in terms of area.

Related Standards: N.VM.1, N.VM.11

Standards for Mathematical Practice:

SMP 4 – Model with mathematics

The Teaching Cycle:

Launch (Whole Class):

Read the initial context with students and verify that they can explain the relationship of the vector diagram to the context. Then set them to work on the six problems.

Explore (Small Group):

Monitor students to observe how they approach each problem. Use the following hints to prompt students who are struggling with particular problems.

Question 1: The wind's speed and the plane's speed are represented by the length of the vectors. How can we find the length of the vectors (i.,e, the distance from the point at the tail to the point at the head of the vector)?

Question 2: The ground speed and actual direction of motion of the plane is represented by the sum of the vectors representing the wind's speed and the plane's speed. Note that the magnitude of the sum of the two vectors is not the sum of the magnitudes of the individual speeds.

Question 3: Why does the diagonal of a parallelogram represent the sum of the vectors?

Question 4: The coordinates of the parallelogram's vertices can by represented by the columns of a matrix. Remind students that we have already found the entries of matrices that reflect images across the x- or y-axes, as well as matrices that rotate images 90°, 180° or 270° about the origin.

Question 5: Given the coordinates of the reflected image based on the matrix multiplication in question 4, how might we determine that opposite sides of this image quadrilateral are parallel?

Question 6: We have found a relationship between the determinant of a matrix and the area of a parallelogram. How might we apply that relationship to this situation? Note: Because this parallelogram is also a rectangle, students might also find the area of the parallelogram using the lengths of the sides, as found in question 1. It would be good to present both approaches to the whole class to confirm that the determinant does find the area of the parallelogram correctly.

Discuss (Whole Class):

Since this is a practice task, you will need to determine what issues have come up in the individual work of the students that might benefit from a whole class discussion. This practice task includes a variety of procedural work including finding distance between points, verifying that line segments are parallel or perpendicular, adding vectors, and multiplying matrices where one factor represents a transformation and the other factor represents the vertices of an image. Make sure that students are confident with carrying out the procedural work, while also interpreting the meaning of the work in terms of the context of the airplane or the context of the geometry of the plane.

Aligned Ready, Set, Go: Connecting Algebra and Geometry 6.14H

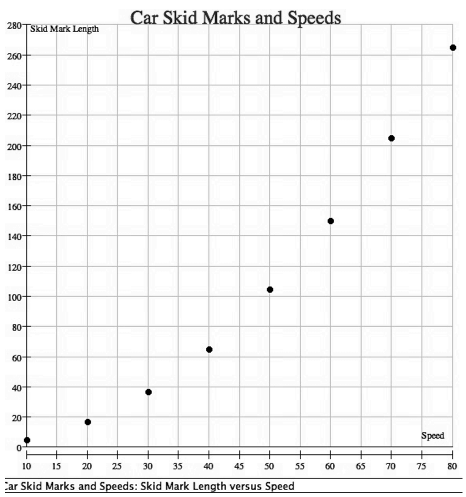
READY, SET, GO! Name Period Date

READY

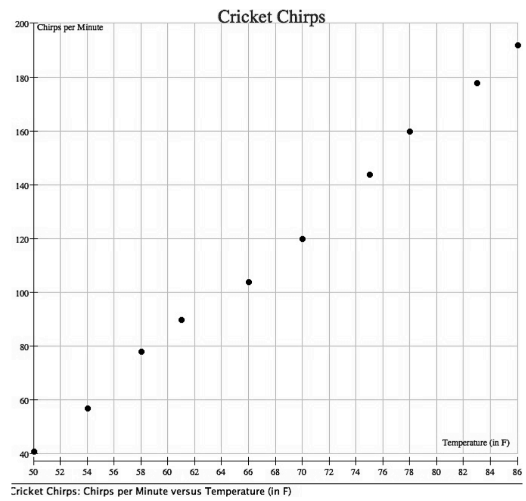
Topic: Scatter Plots and Trend Lines

Examine each of the scatterplots shown below. If possible, make a statement about relationships between the two quantities depicted in the scatterplot.

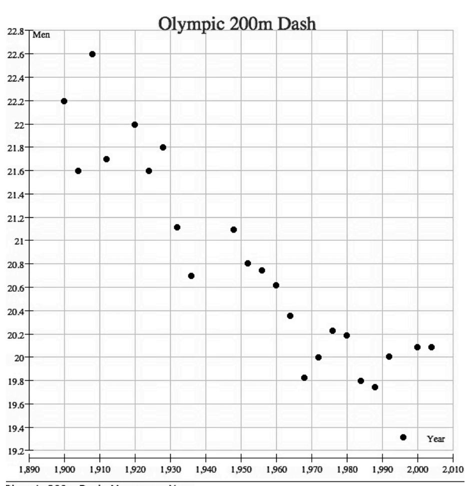
1.



2.



3.



4. For each scatterplot, write the equation of a trend line that you think best fits the data.

a. Trend line #1

b. Trend line #2

c. Trend line #3

SET

Topic: Applications of vectors.

Given: $\vec{u}:\langle -5,1\rangle$, $\vec{v}:\langle 3,5\rangle$, $\vec{w}:\langle 4,-3\rangle$. **Each of these three vectors represents a force pulling on an object—such as in a three-way tug of war—with force exerted in each direction being measured in pounds.**

5. Find the magnitude of each vector. That is, how many pounds of force are being exerted on the object by each tug? (Round to the nearest hundredth)

a. $\|\vec{u}\| =$

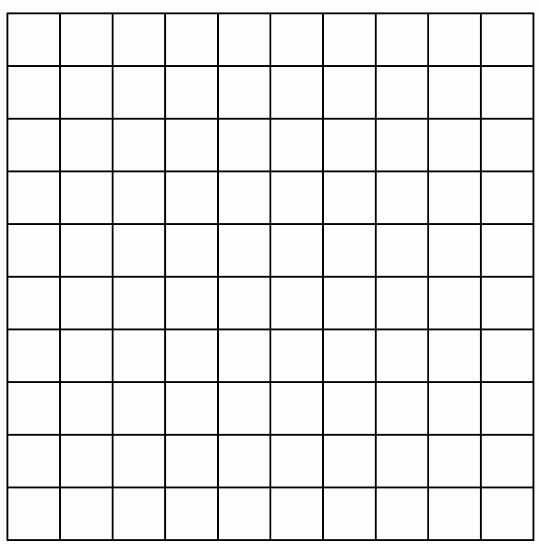
b. $\|\vec{v}\| =$

c. $\|\vec{w}\| =$

6. Find the magnitude of the sum of the three forces on the object.

$\|\vec{u} + \vec{v} + \vec{w}\| =$

7. Draw a vector diagram showing the resultant direction and magnitude of the motion resulting from this three-way tug of war.



GO

Topic: Solving Systems

8. Solve the given system in each of the following ways.

Given: $\begin{cases} 4x - 4y = 7 \\ 6x - 8y = 9 \end{cases}$

a. By substitution

b. By elimination

c. Using matrix row reduction

d. Using an inverse matrix