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#35 Multiple Representations and Perseverance

A Tool for When the Going Gets Tough

Presenter: Janet Sutorius

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Learning is a consequence of thinking.

Learning mathematics is a consequence of:

- The mathematical **thinking** required by the problems that are presented to students,
- The **thinking** students do while representing the work of the problem, and
- The **thinking** students do as they make connections between the various representations of the problem.

The 8 Standards for Mathematical Practice

Mathematically proficient students . . .

1. Make sense of problems and persevere in solving them.
2. Reason abstractly and quantitatively.
3. Construct viable arguments and critique the reasoning of others.
4. Model with mathematics.
5. Use appropriate tools strategically.
6. Attend to precision.
7. Look for and make use of structure.
8. Look for and express regularity in repeated reasoning.

Mathematically proficient students …
(keep in mind that not all of our students are mathematically proficient!)

Make sense of problems and persevere in solving them.

- Make sense of the meaning of quantities and their relationships in problem situations.
- Look for entry points.
- Use representations to help conceptualize problems and search for regularities.
- Explain correspondences between representations.

NCTM'S Effective Teaching Practices (2014 Principles to Action)

Mathematics Teaching Practices
Establish mathematics goals to focus learning. Effective teaching of mathematics establishes clear goals for the mathematics that students are learning, situates goals within learning progressions, and uses the goals to guide instructional decisions.
Implement tasks that promote reasoning and problem solving. Effective teaching of mathematics engages students in solving and discussing tasks that promote mathematical reasoning and problem solving and allow multiple entry points and varied solution strategies.
Use and connect mathematical representations. Effective teaching of mathematics engages students in making connections among mathematical representations to deepen understanding of mathematics concepts and procedures and as tools for problem solving.
Facilitate meaningful mathematical discourse. Effective teaching of mathematics facilitates discourse among students to build shared understanding of mathematical ideas by analyzing and comparing student approaches and arguments.
Pose purposeful questions. Effective teaching of mathematics uses purposeful questions to assess and advance students' reasoning and sense making about important mathematical ideas and relationships.
Build procedural fluency from conceptual understanding. Effective teaching of mathematics builds fluency with procedures on a foundation of conceptual understanding so that students, over time, become skillful in using procedures flexibly as they solve contextual and mathematical problems.
Support productive struggle in learning mathematics. Effective teaching of mathematics consistently provides students, individually and collectively, with opportunities and supports to engage in productive struggle as they grapple with mathematical ideas and relationships.
Elicit and use evidence of student thinking. Effective teaching of mathematics uses evidence of student thinking to assess progress toward mathematical understanding and to adjust instruction continually in ways that support and extend learning.

NCTM is telling us to Support Productive Struggle in Learning Mathematics

Effective teaching of mathematics consistently provides students with opportunities and supports to engage in productive struggle as they grapple with mathematical ideas and relationships.

from Principles to Action NCTM 2014

What is the difference between productive struggle and unproductive struggle?

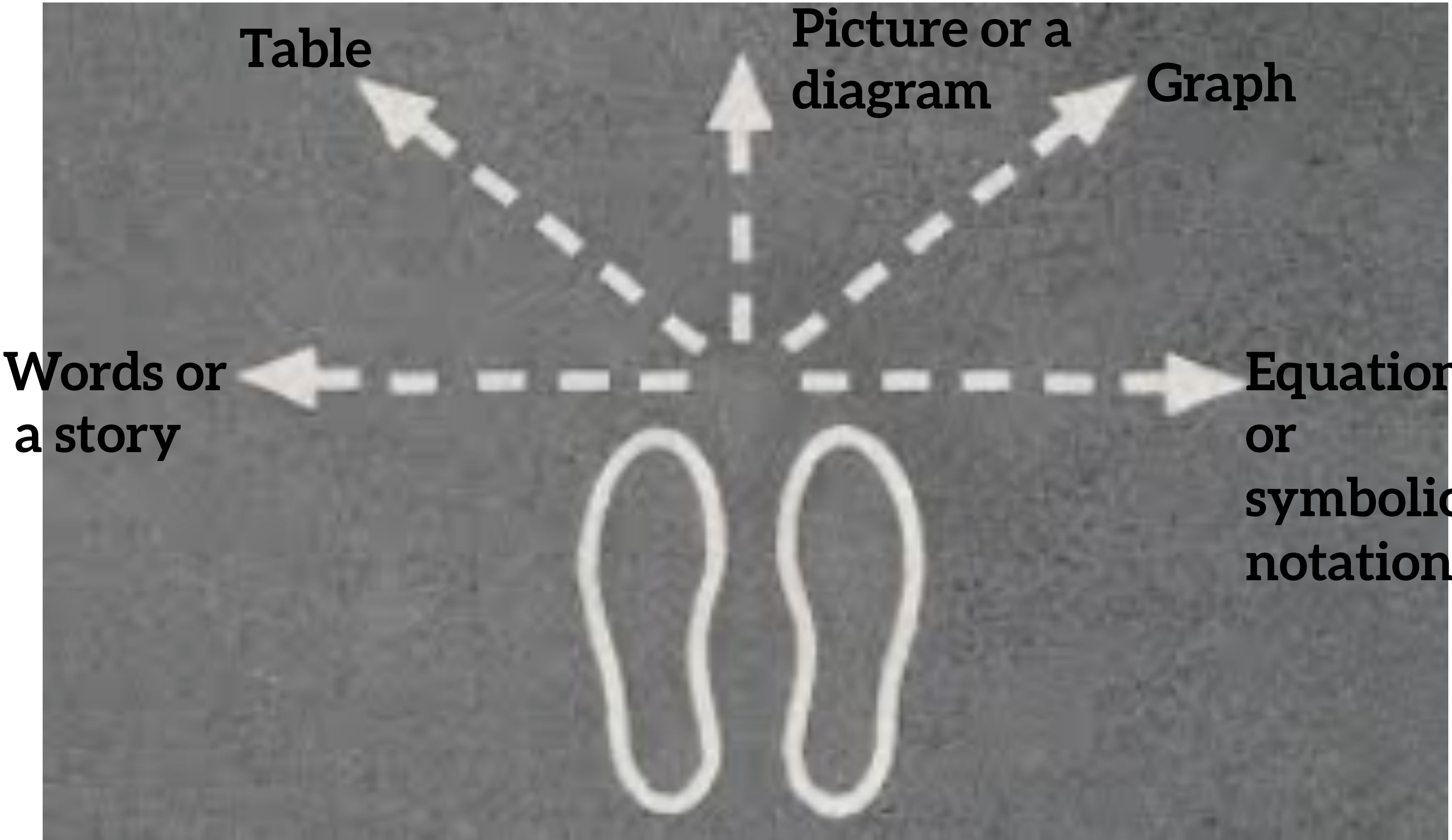
In contrast to productive struggle, ***unproductive struggle*** ***occurs when students "make no progress towards sense-making, explaining, or proceeding with a problem or task at hand."*** *(Warshauer 2011, p.21) Principles to Action NCTM 2014 page 48*

Sometimes students don’t know how to move forward.

No one has taught them how to think mathematically.

They are stuck and don’t even know how to begin.

We can start with representations



Why should perseverance be linked with multiple representations?

- Representations provide access points for students who don’t know how to begin thinking about the problem.
- Representations are the vehicles through which students explore content.
- Connecting representations moves students’ intuitive thinking towards conventional and efficient mathematics.

Problem 1:
How many bows that are 5/12 yards long can be made from 3 yards of ribbon?

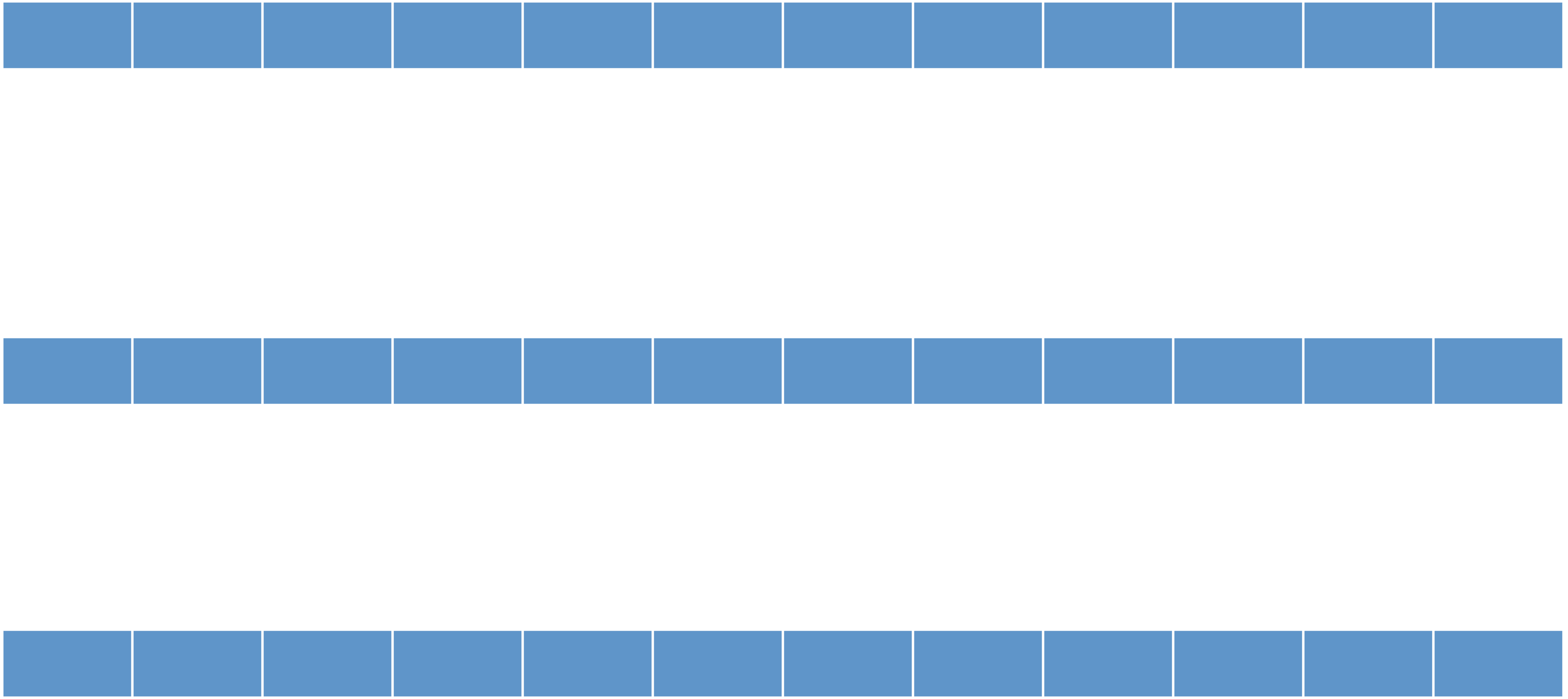
$$7\frac{1}{5}$$

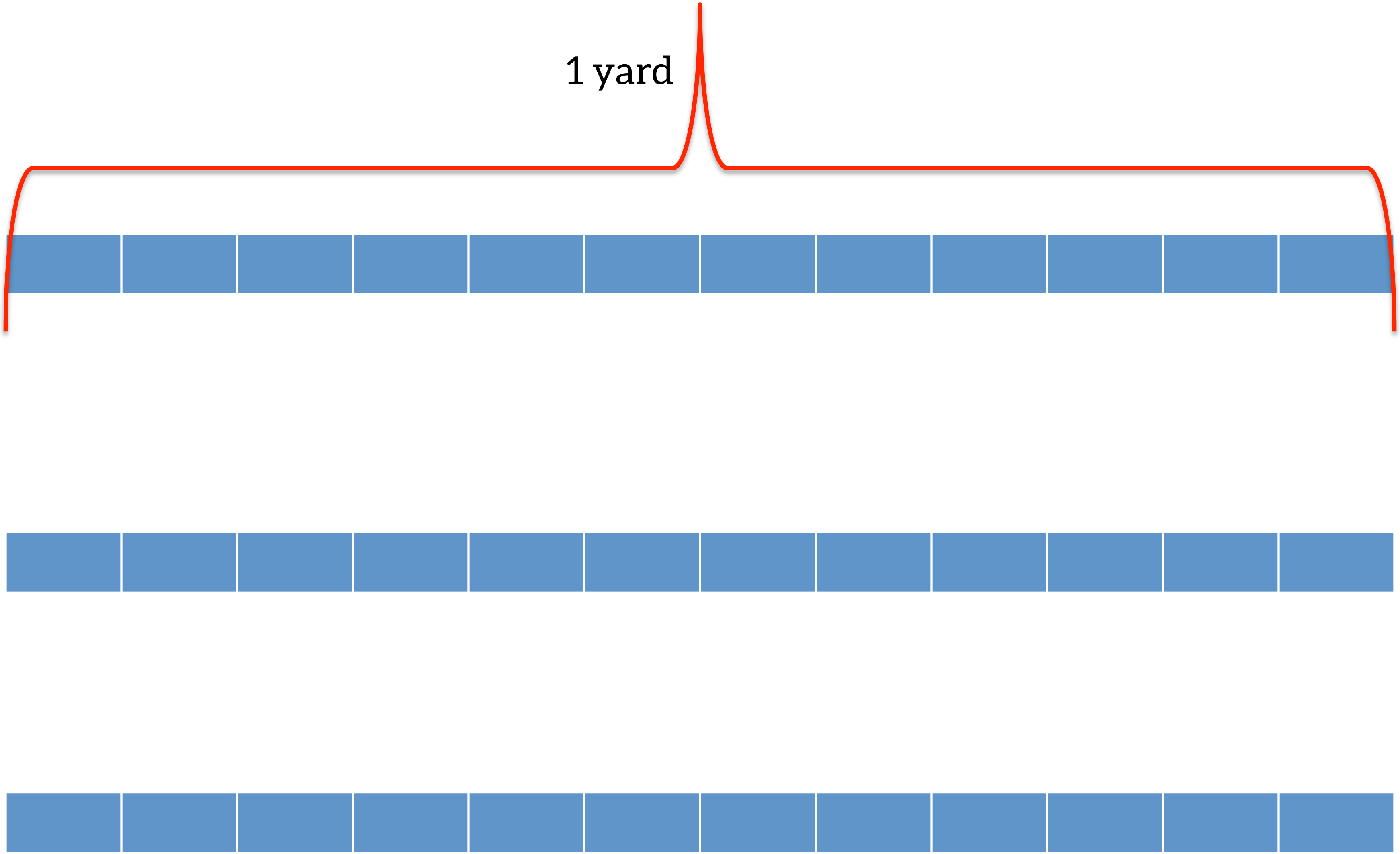
$$7\frac{1}{12}$$

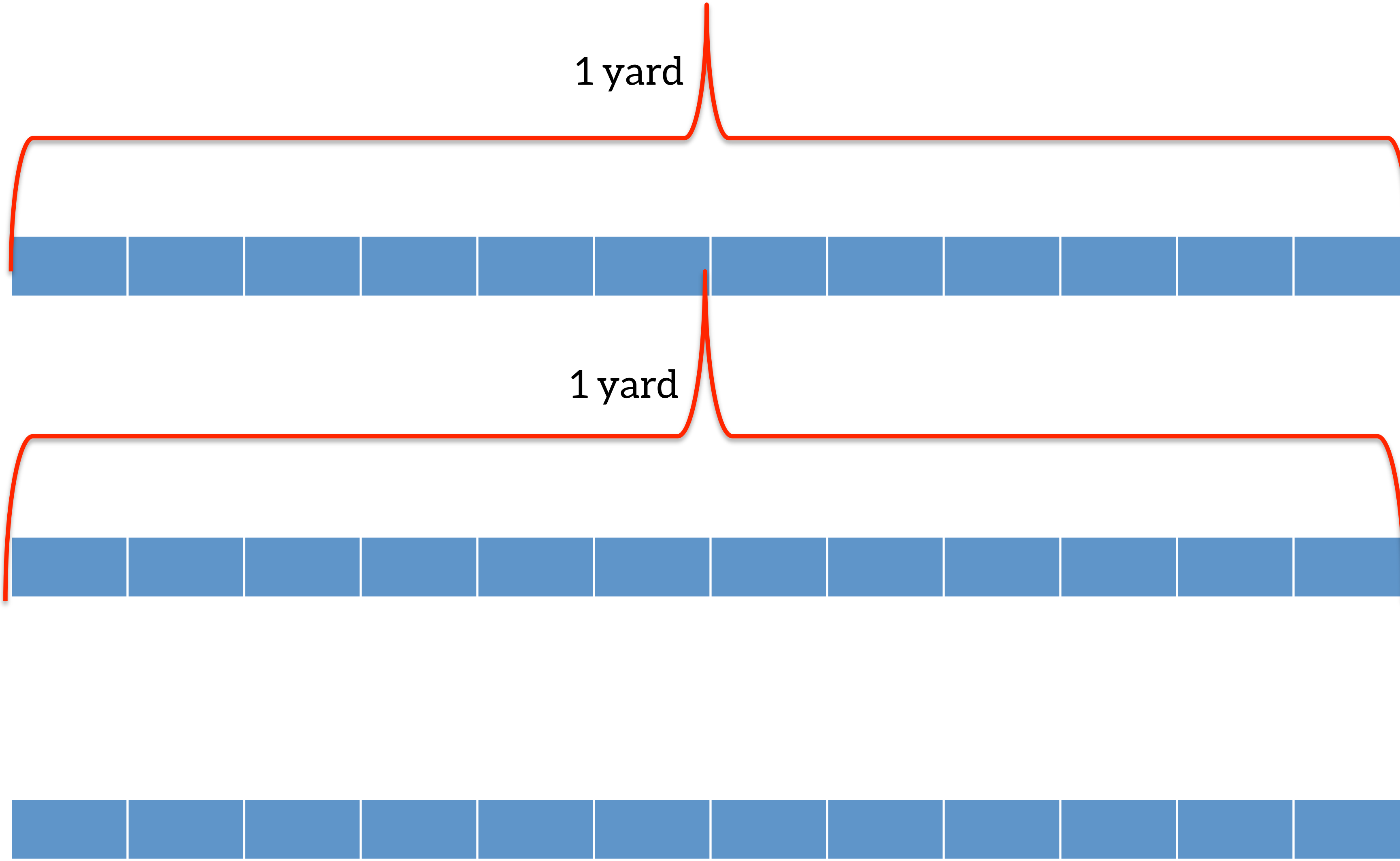
I knew that we were all doing something right because we all got 7 for the number of bows.

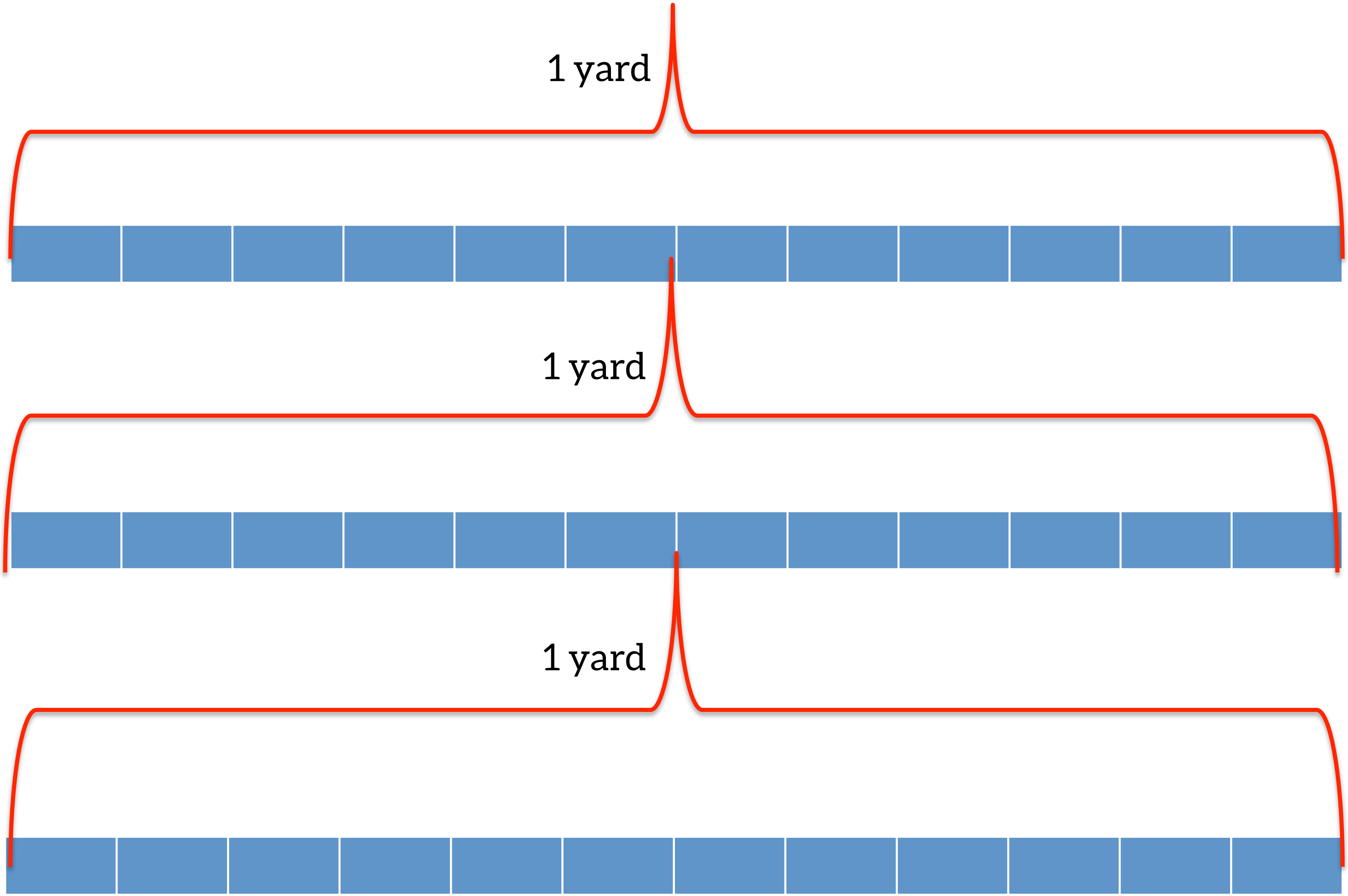
But what was going on with the fraction?

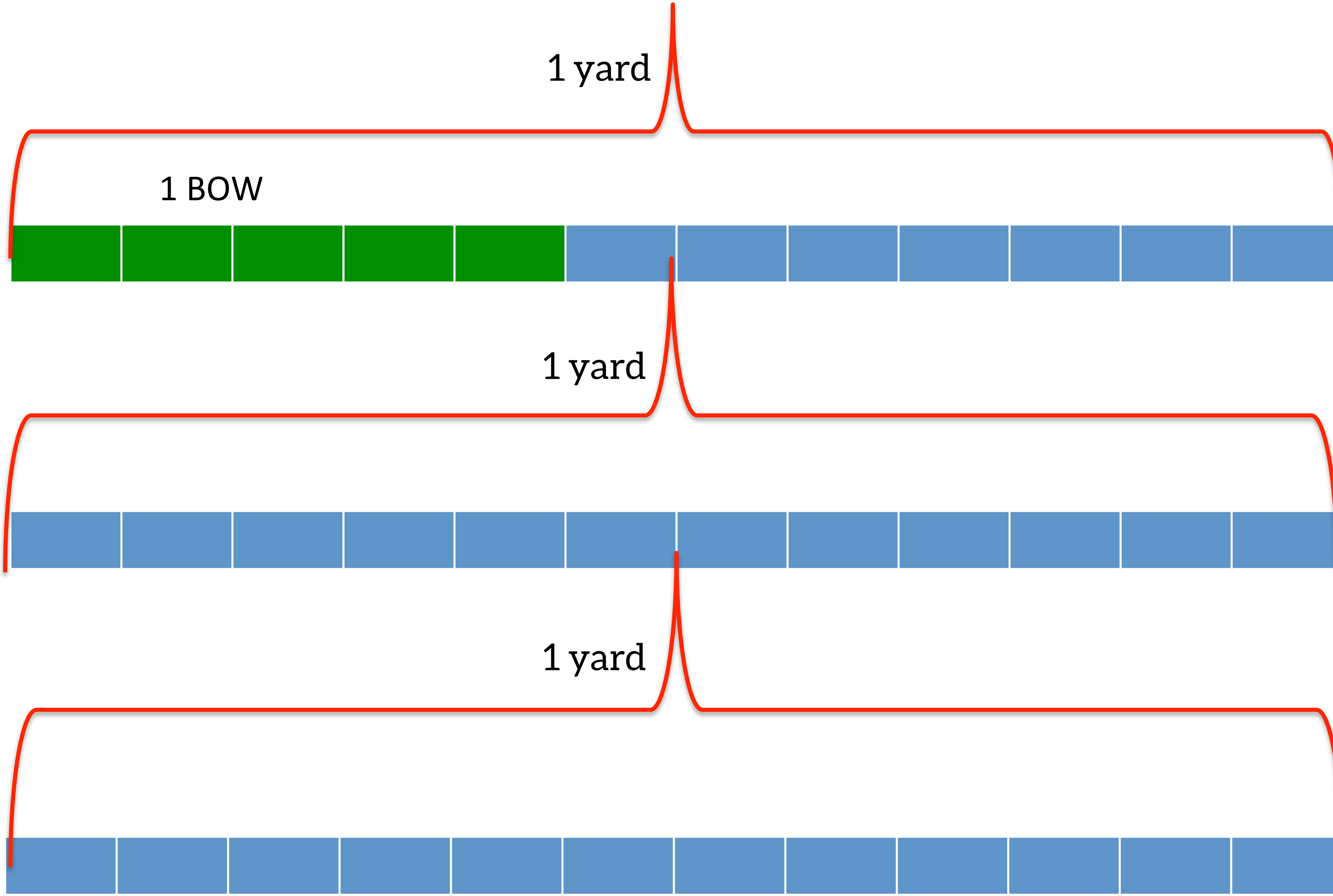
1. Did anyone draw a diagram to help them think about the problem?

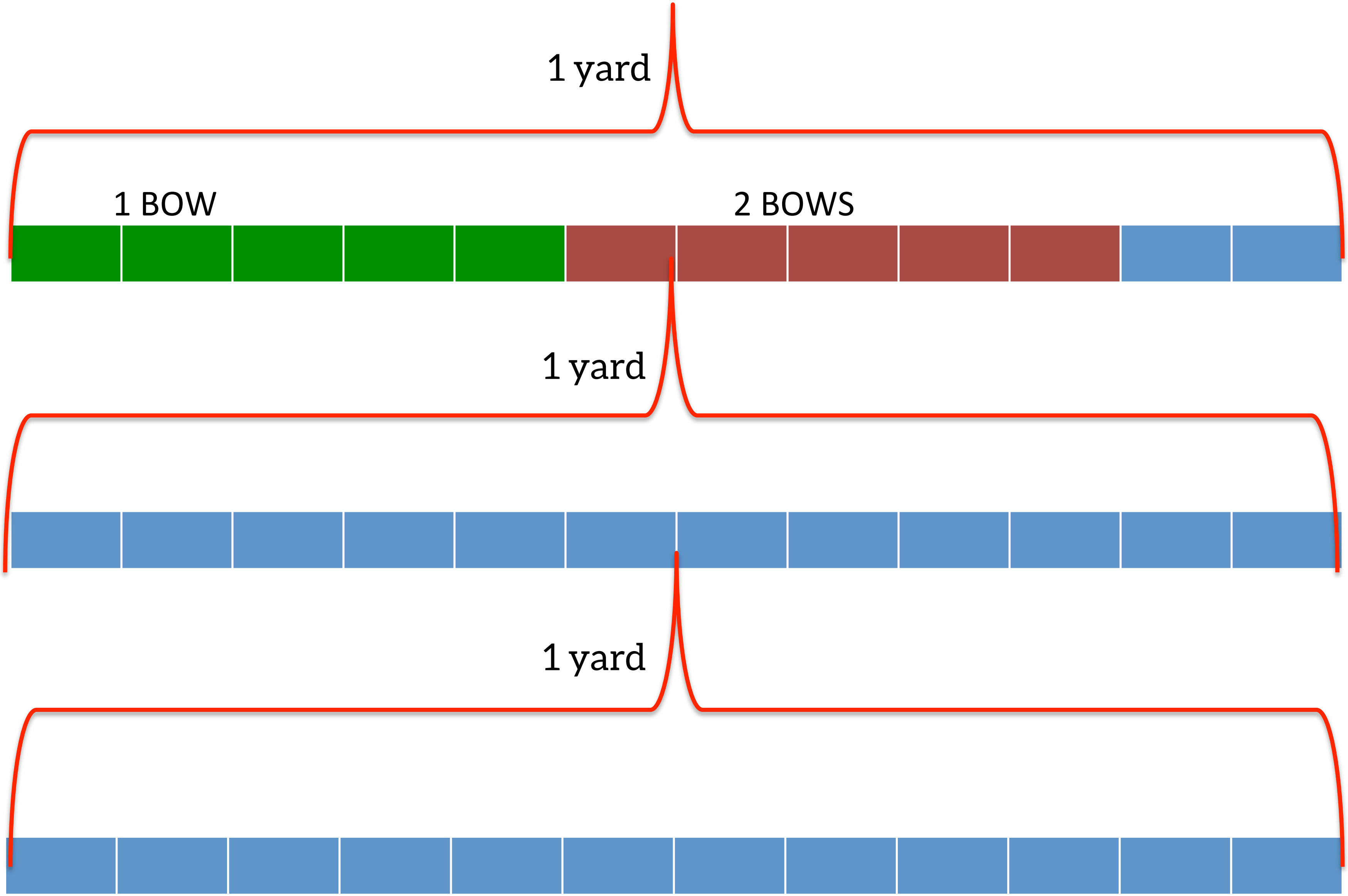


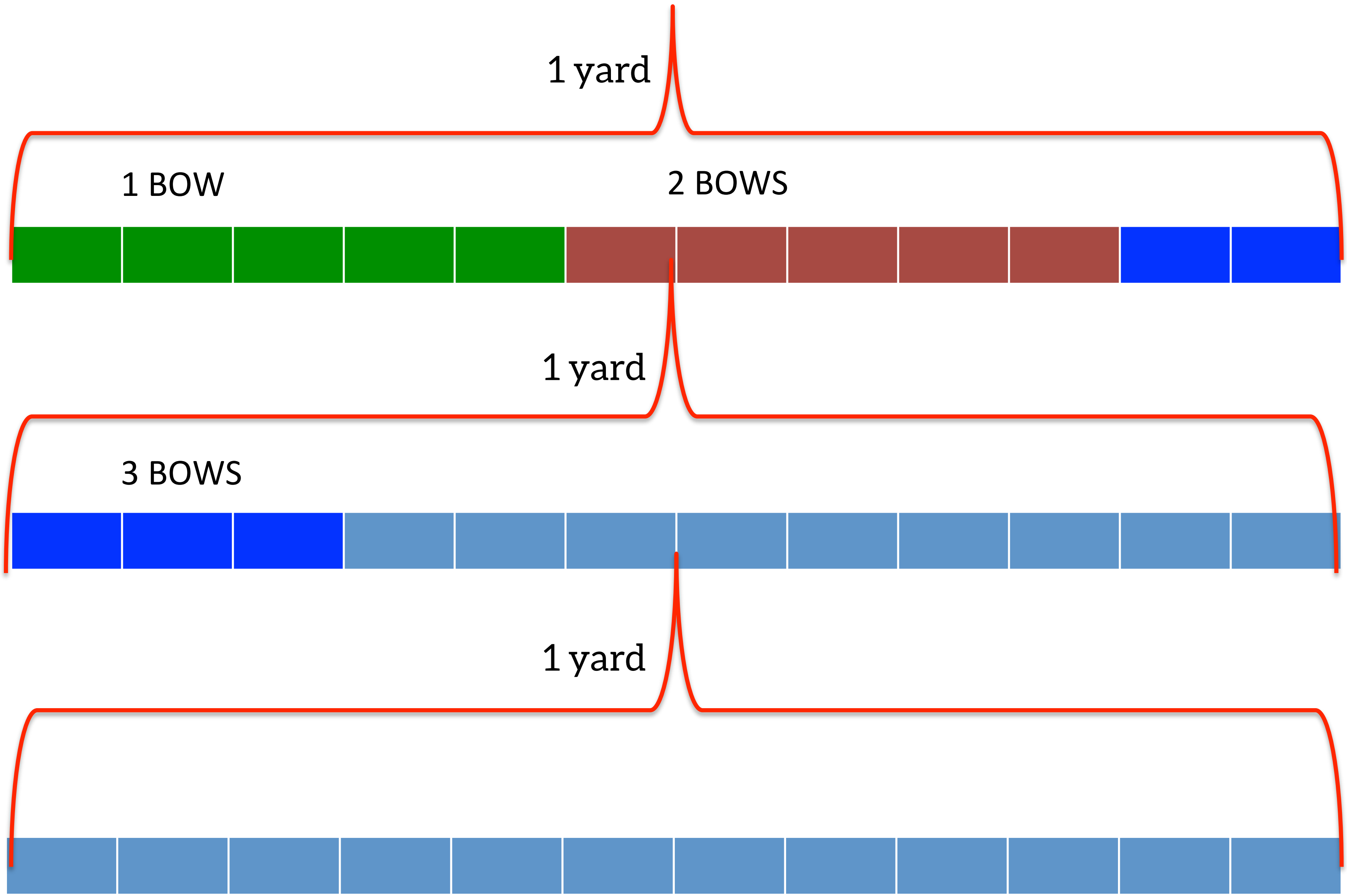

1 yard

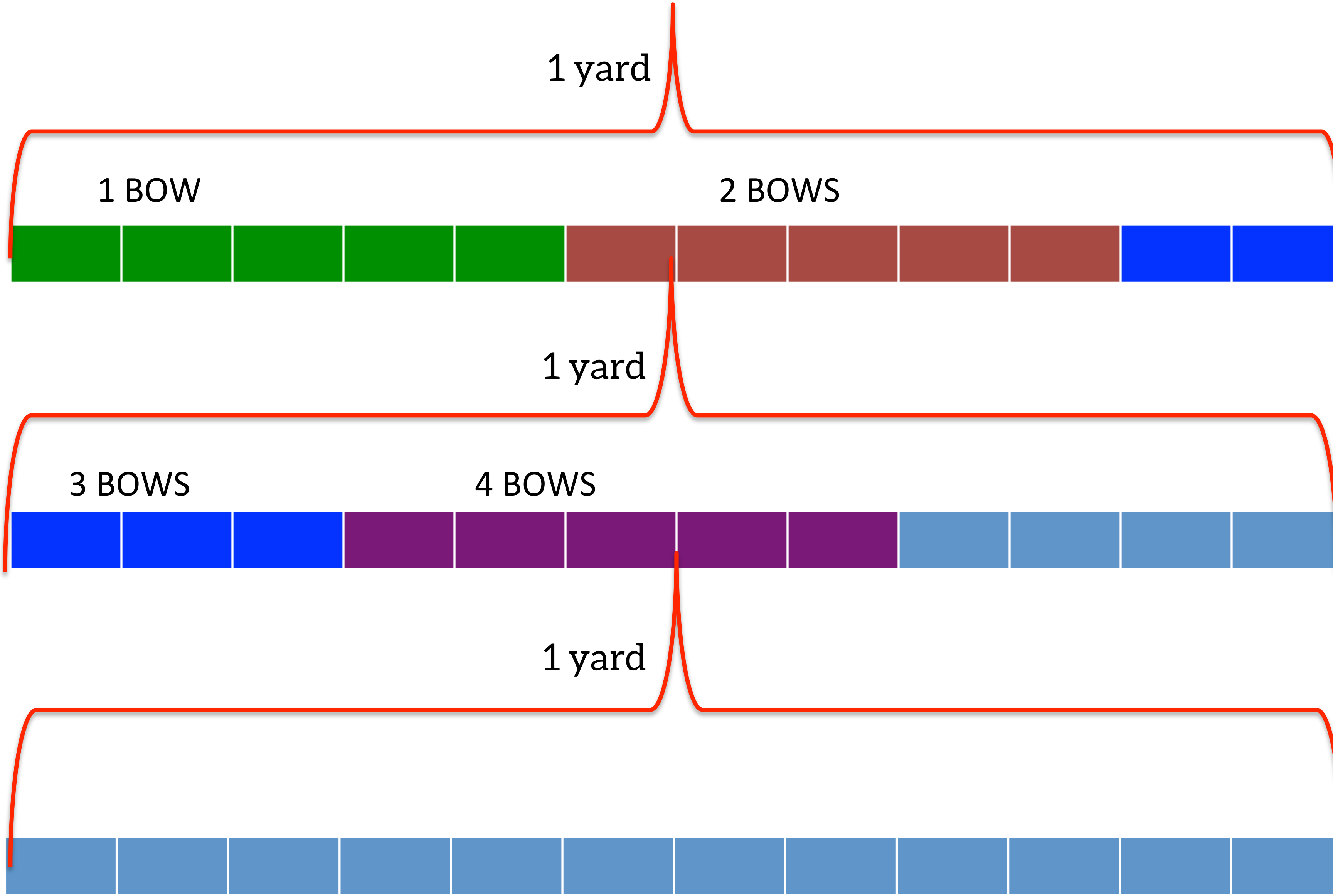

1 yard
1 yard

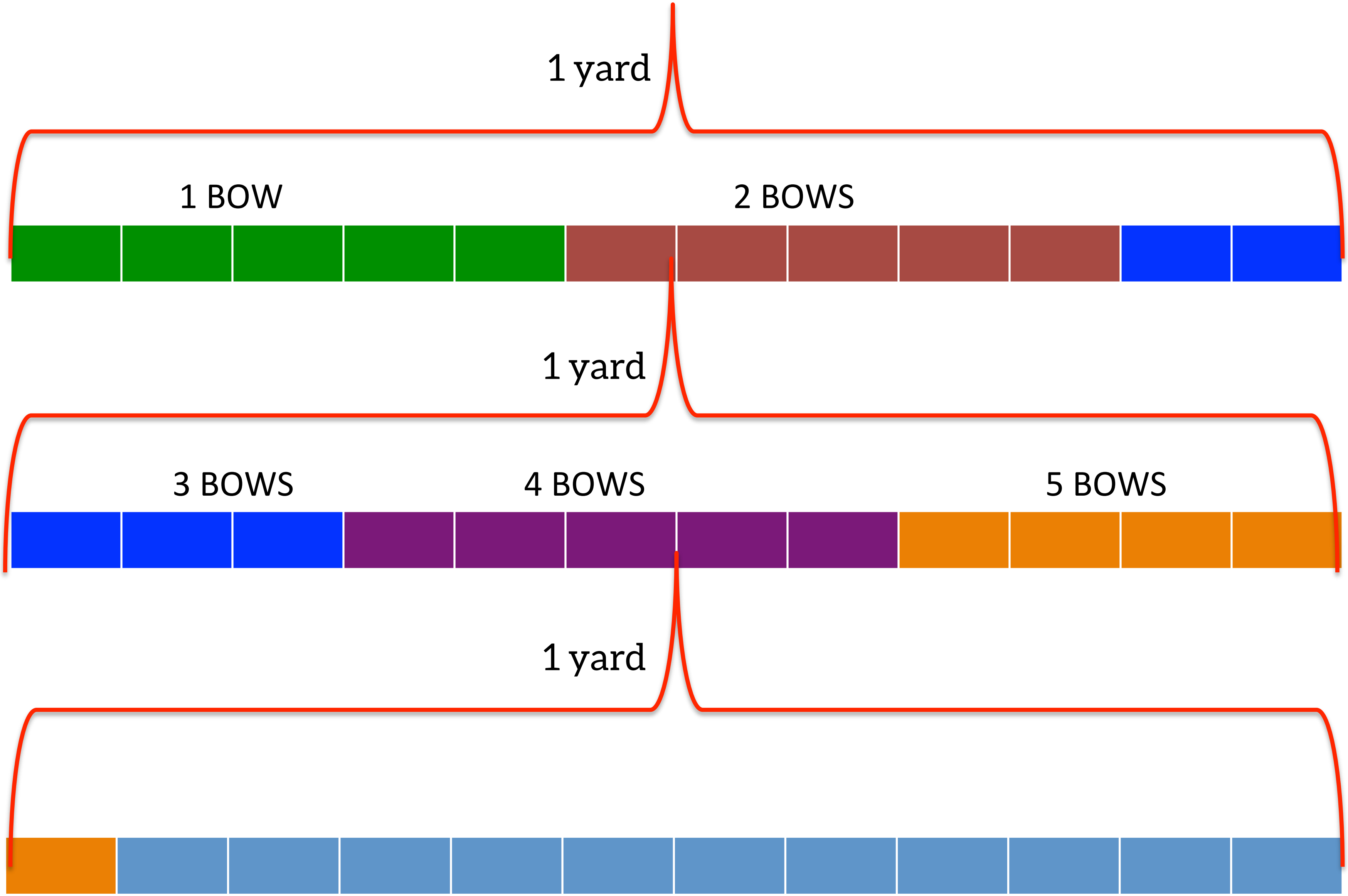

1 yard
1 yard
1 yard

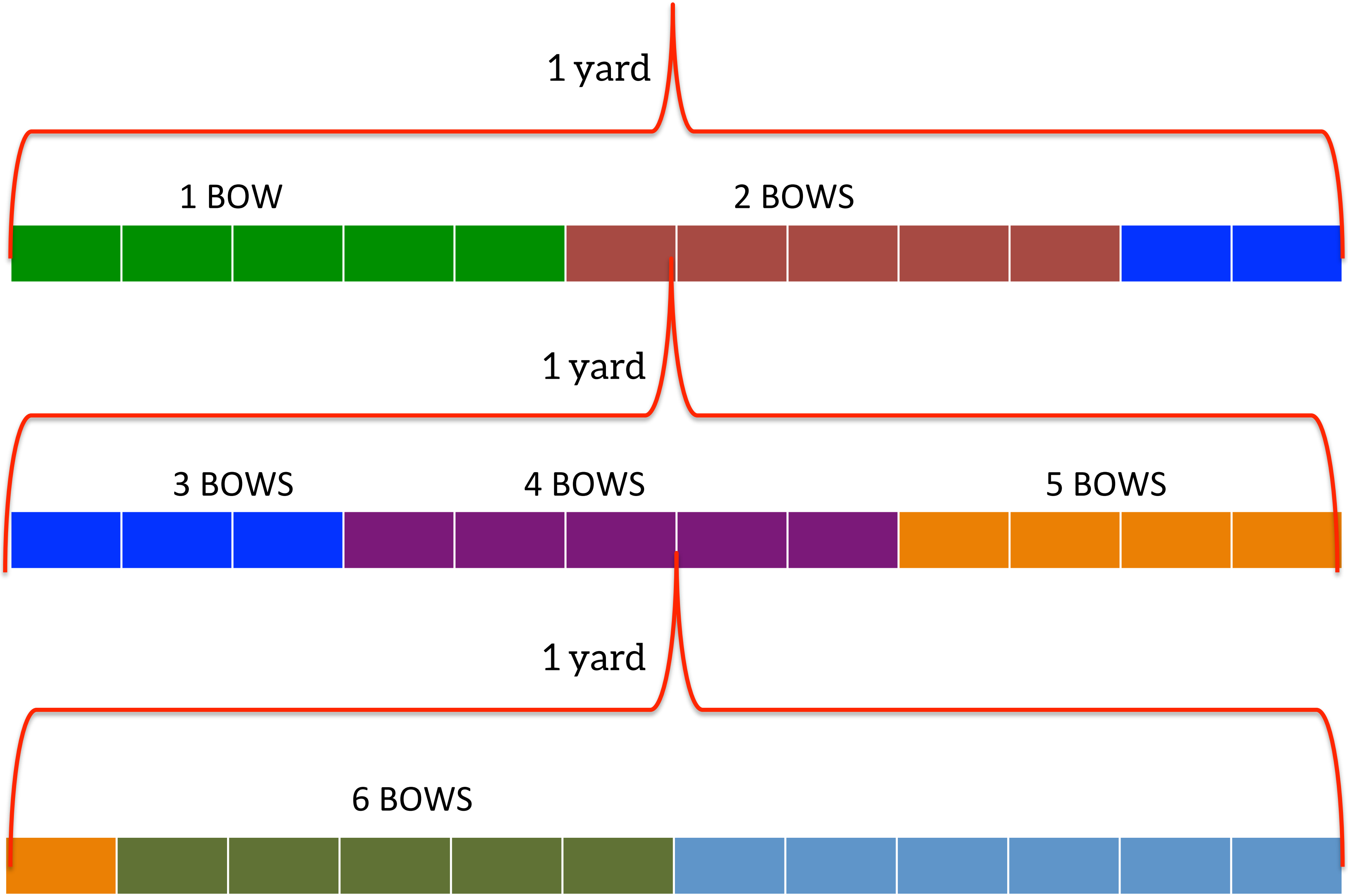

1 yard
1 BOW
1 yard
1 yard


1 yard
1 BOW
2 BOWS
1 yard
1 yard

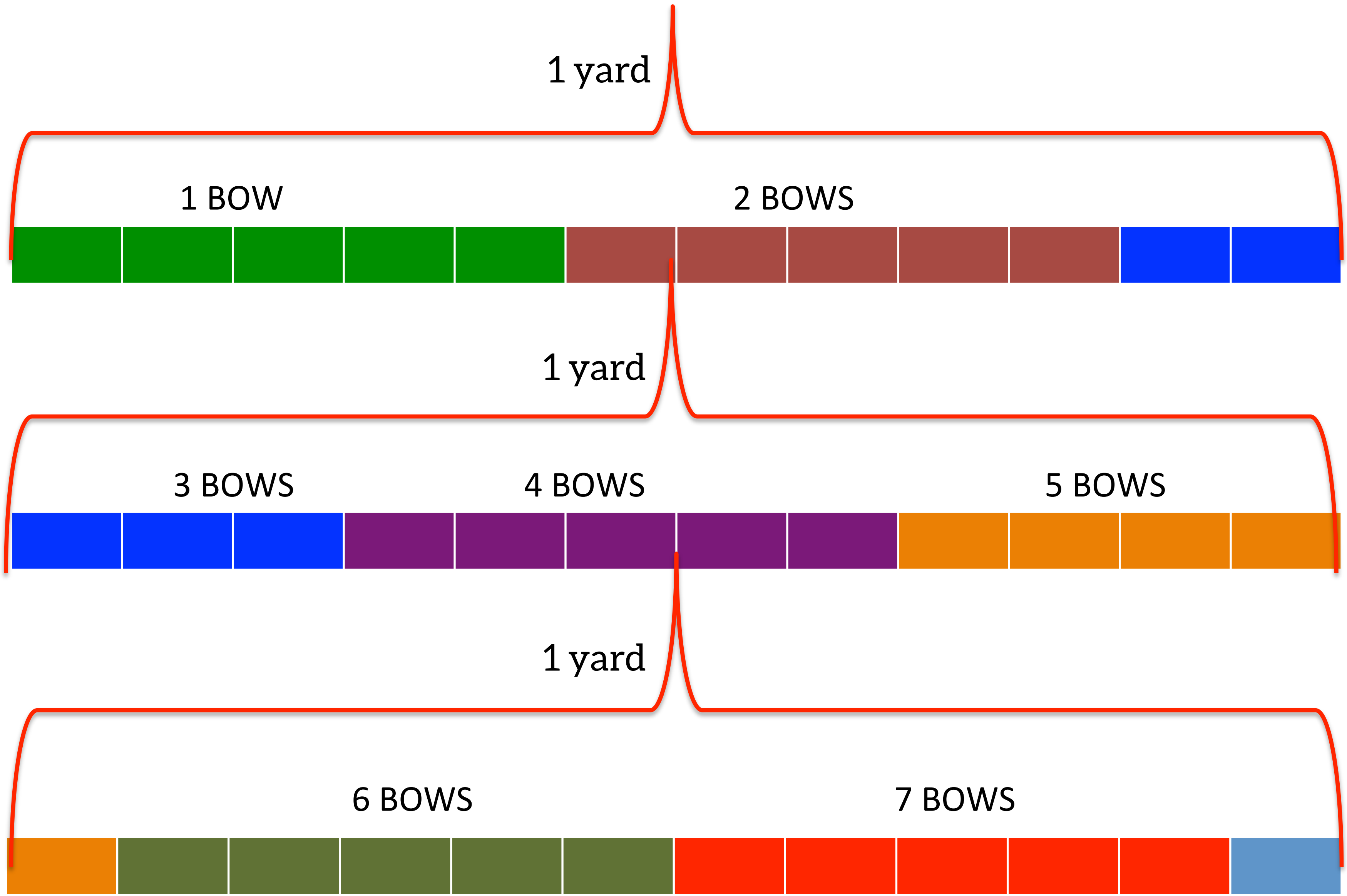

1 yard
1 BOW
2 BOWS
1 yard
3 BOWS
1 yard

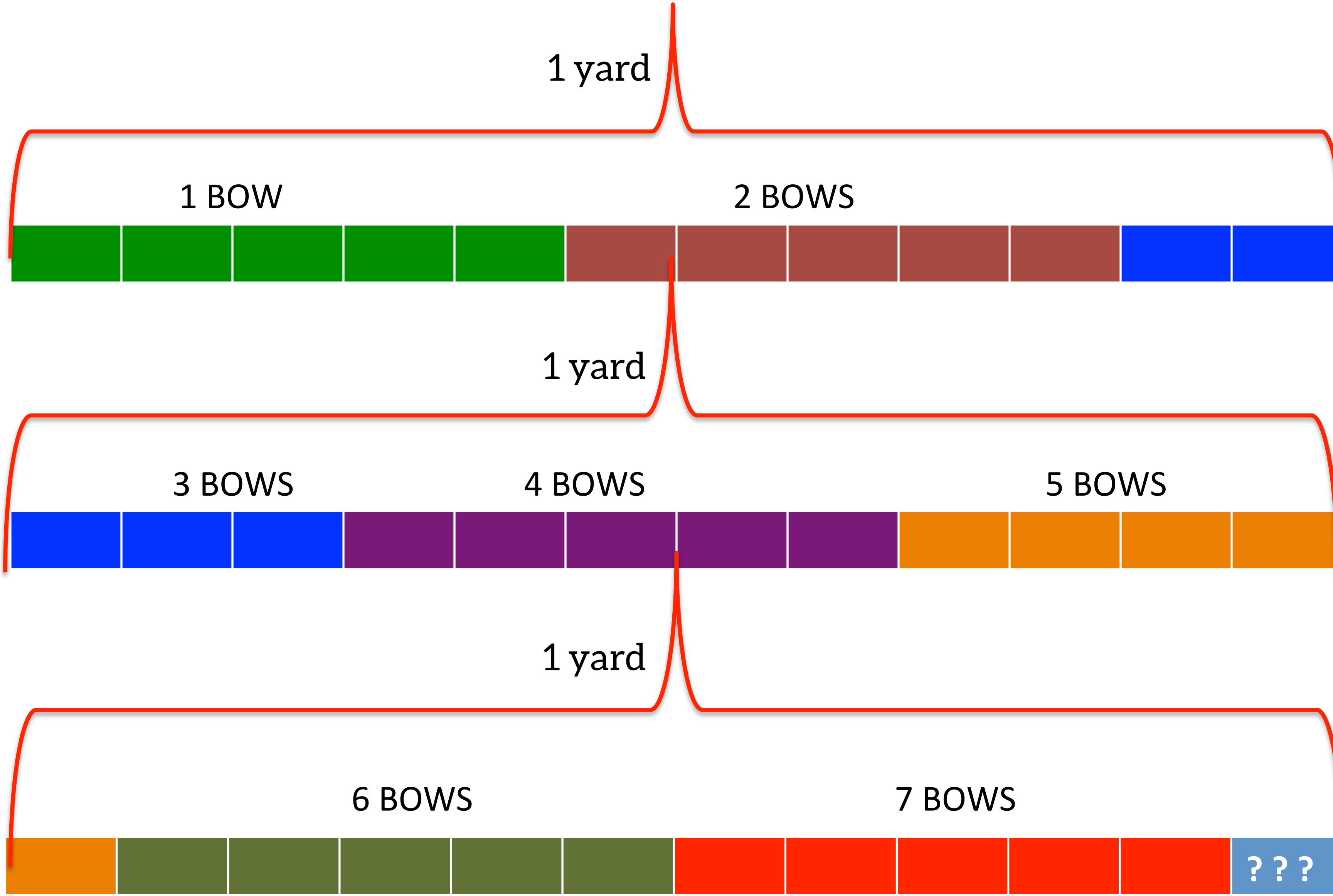

1 yard
1 BOW
2 BOWS
1 yard
3 BOWS
4 BOWS
1 yard


1 yard
1 BOW
2 BOWS
1 yard
3 BOWS
4 BOWS
5 BOWS
1 yard



1 yard
1 BOW
2 BOWS
1 yard
3 BOWS
4 BOWS
5 BOWS
1 yard
6 BOWS


1 yard
1 BOW
2 BOWS
1 yard
3 BOWS
4 BOWS
5 BOWS
1 yard
6 BOWS
7 BOWS


1 yard
1 BOW
2 BOWS
1 yard
3 BOWS
4 BOWS
5 BOWS
1 yard
6 BOWS
7 BOWS
? ? ?

Does anyone still have a question?

Student, "I thought about the problem differently …"

I know each yard contains 36 inches.
I have 3 yards so that's 108 inches total.
1 bow is 5/12 of a yard. 5/12 times 36 = 15 inches

Number of bows	Inches remaining
0	108 (subtract 15)
1	93 (subtract 15)
2	78 (subtract 15)
3	63 (subtract 15)
4	48 (subtract 15)
5	33 (subtract 15)
6	18 (subtract 15)
7	3 inches leftover

What does the 3 inches represent?

Number of bows	Inches remaining
0	108 (subtract 15)
1	93 (subtract 15)
2	78 (subtract 15)
3	63 (subtract 15)
4	48 (subtract 15)
5	33 (subtract 15)
6	18 (subtract 15)
7	3 inches leftover

1/4 of a foot

1/12 of a yard

1/5 of a bow

What can we learn from this story?

The naked mathematics of procedures and answer getting can hide the excellent mathematical thinking that students are doing.

The sharing of representations can deepen everyone's understanding of a concept.

Students can think differently from their neighbor and the teacher and still be correct in their thinking.

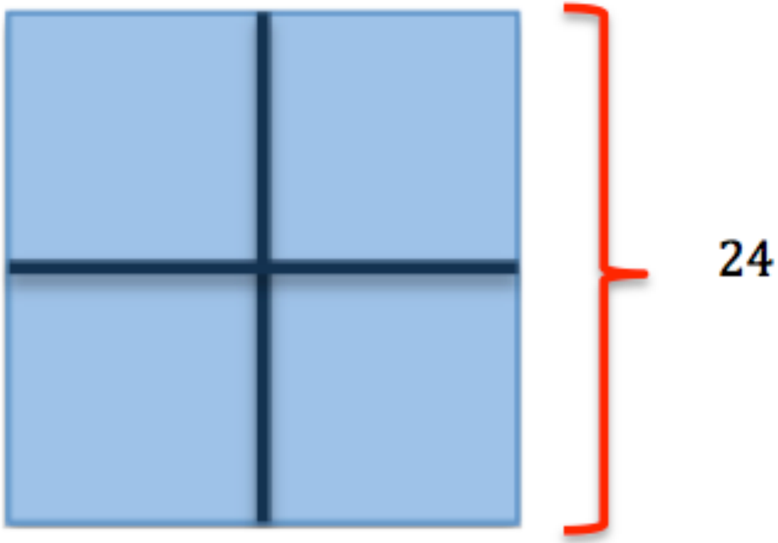
The students and teacher can learn from each other.

Problem 2
4.1 Cafeteria Actions and Reactions

Elvira, the cafeteria manager, has just received a shipment of new trays with the school logo prominently displayed in the middle of the tray. After unloading 4 cartons of trays in the pizza line, she realizes that students are arriving for lunch and she will have to wait until lunch is over before unloading the remaining cartons. The new trays are very popular and in just a couple of minutes 24 students have passed through the pizza line and are showing off the school logo on the trays. At this time, Elvira decides to divide the remaining trays in the pizza line into 3 equal groups so she can also place some in the salad line and the sandwich line, hoping to attract students to the other lines. After doing so, she realizes that each of the three serving lines has only 12 of the new trays.

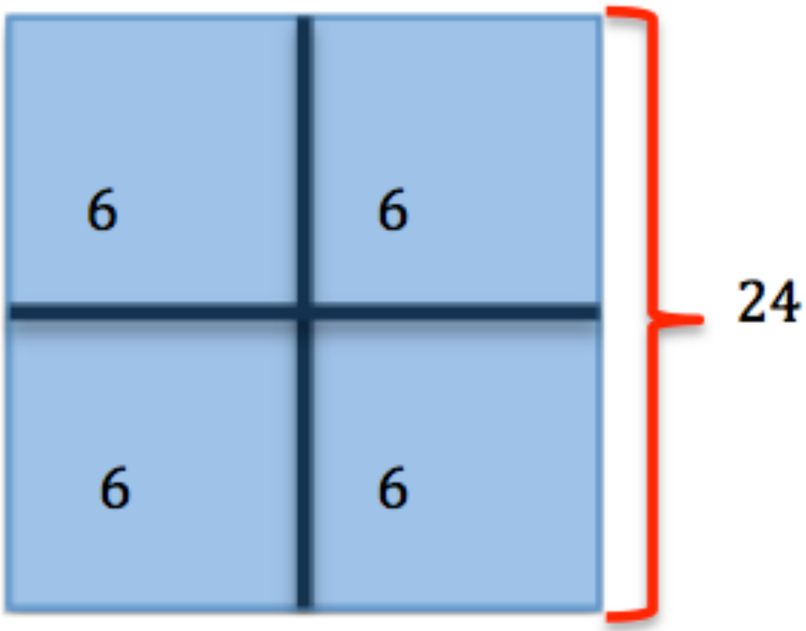
- **"That's not many trays for each line. I wonder how many trays there were in each of the cartons I unloaded?"**

4.1 Cafeteria Actions and Reactions



- I know that Elvira removed 24 trays from the 4 crates.

4.1 Cafeteria Actions and Reactions



- 24 trays were removed from the 4 crates.
- We can think of that as 6 trays having been removed from each crate.

4.1 Cafeteria Actions and Reactions

12

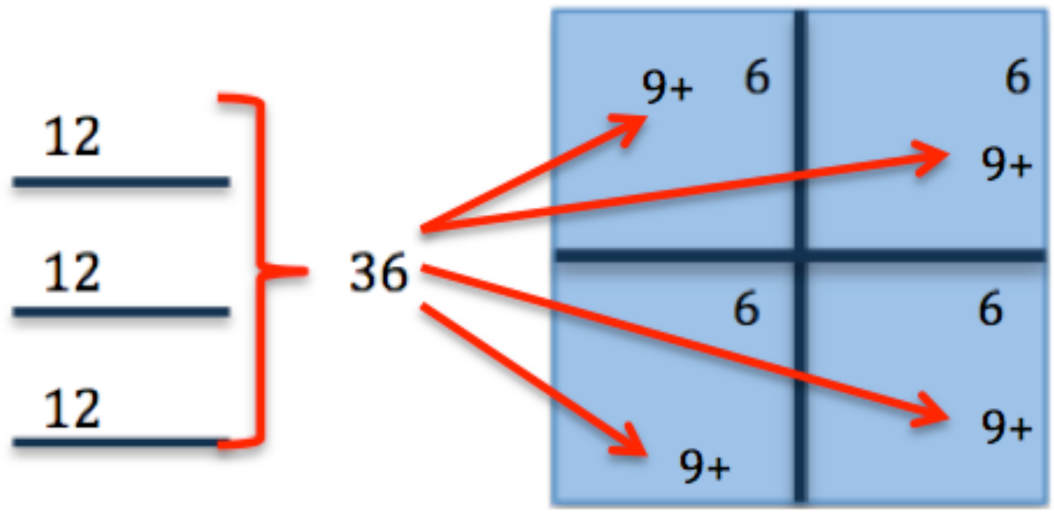
12

12

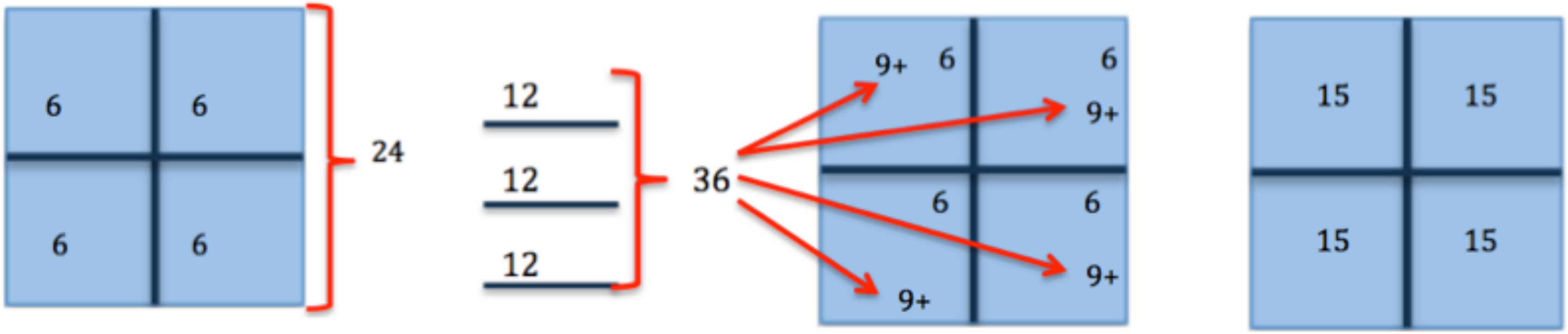
- The remaining trays were divided into 3 lines. That means that there were **36** more trays remaining in the crates.

4.1 Cafeteria Actions and Reactions

- I can divide those 36 trays by 4 and put them back into the 4 crates to figure out how many trays were in the crates at the beginning.

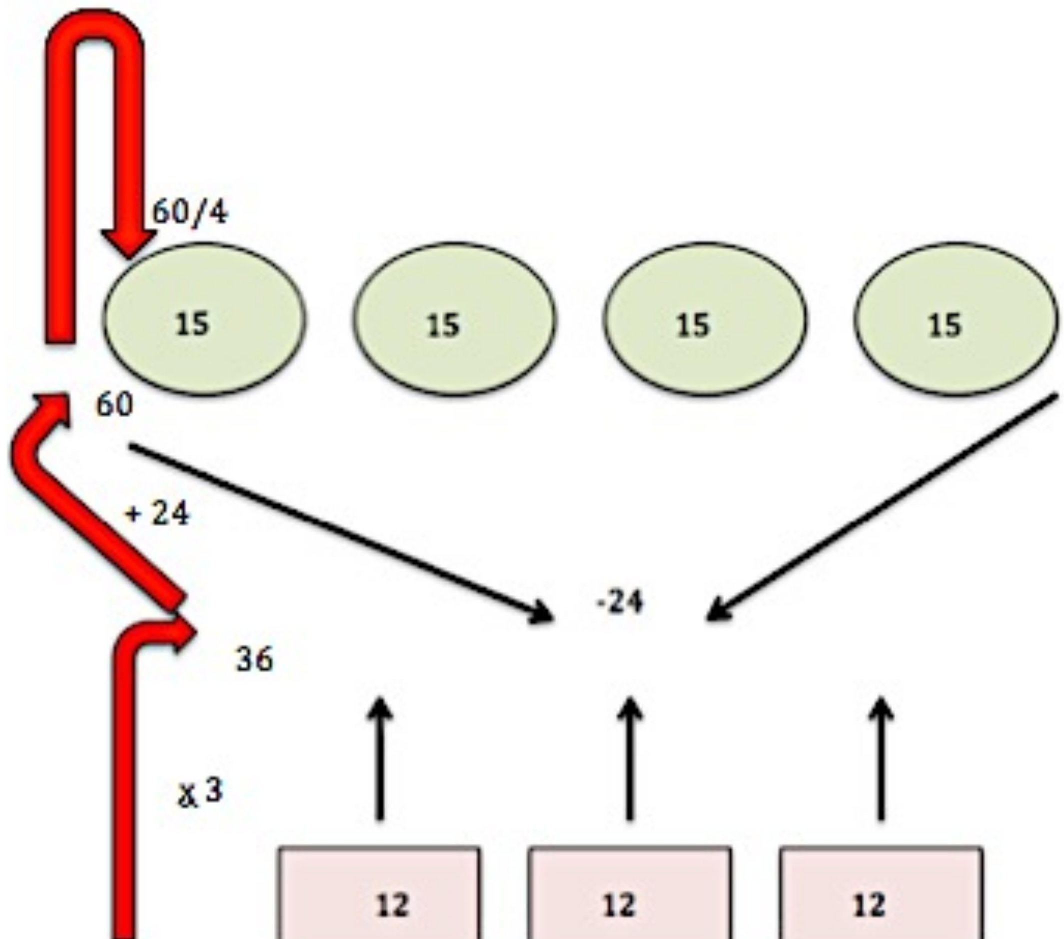


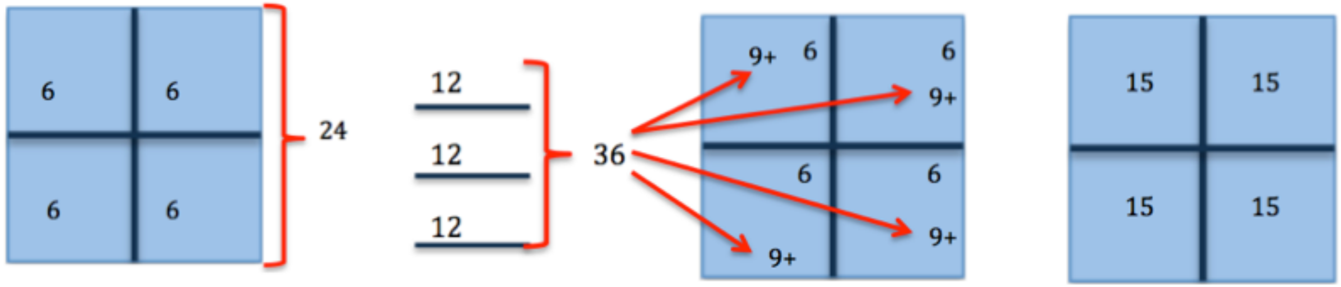
4.1 Cafeteria Actions and Reactions



How is this student's thinking different from the previous student?

I started at the end. I knew I had 12 trays in each of the 3 lines so I multiplied by 3 to get 36. Since I had already taken 24 trays out of the crates I added 24 to 36. That added up to 60 total trays. Then I divided 60 by 4 to find out that I had 15 trays in each crate to begin with.





$$x = 15$$

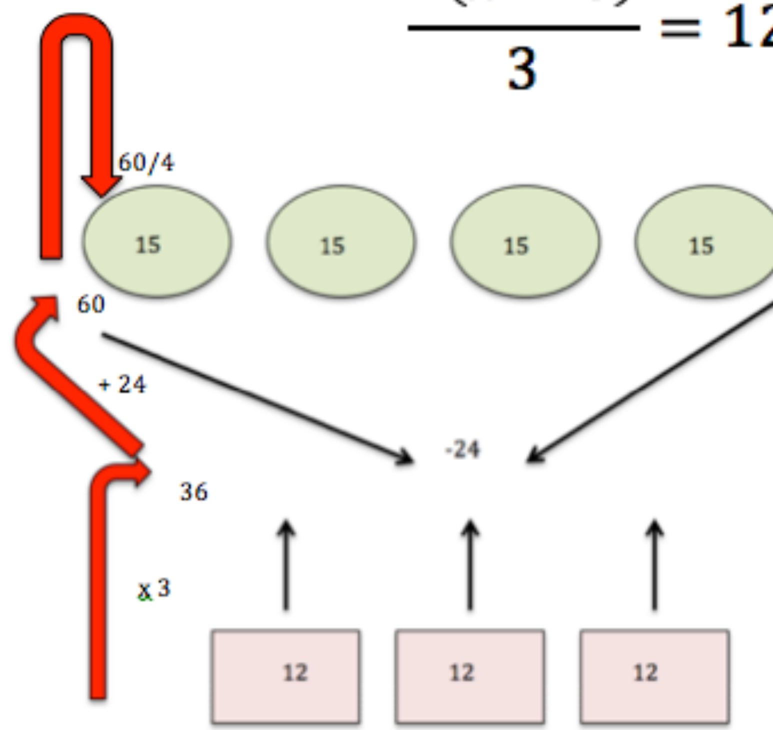
$$4x = 60$$

$$4x - 24 = 36$$

$$\frac{4x - 24}{3} = 12$$

$$\frac{4(x - 6)}{3} = 12$$

$$\frac{4(x - 6)}{3} = 12$$



Problem 3
If $x > y$, then which is greater? $x + a$ or $y + a$

- **Make a convincing argument using a representation such as a diagram, story, or number line to support your answer.**

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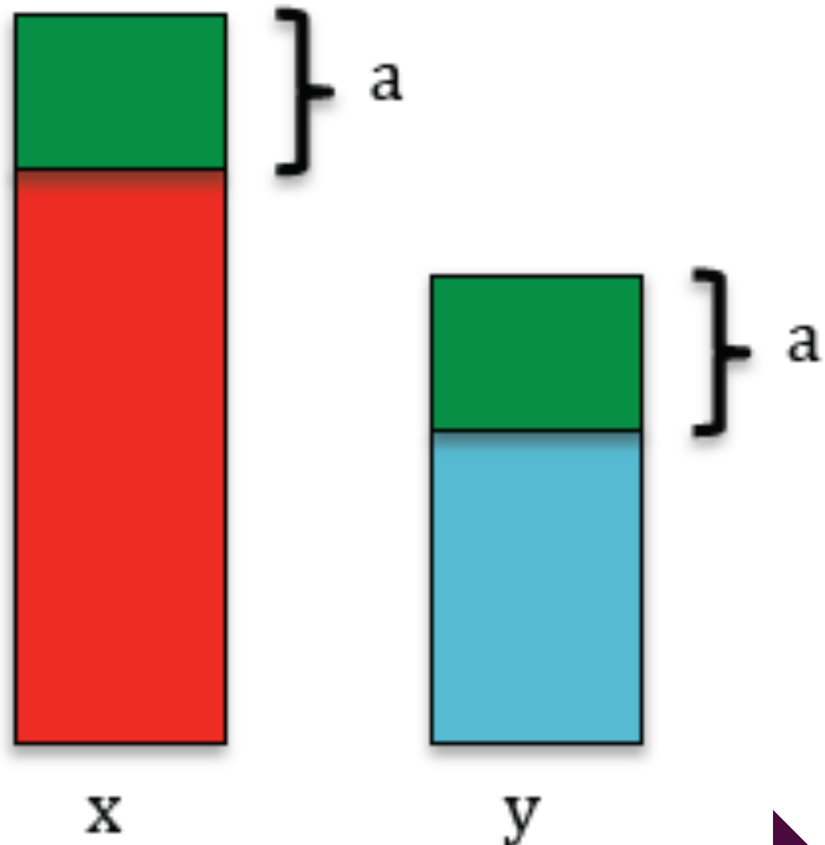
Problem 3
If $x > y$, then which is greater? $x + a$ or $y + a$



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If $x > y$, then which is greater? $x + a$ or $y + a$

- John is older than Marie.
- 10 years from now John will still be older than Marie.
- 2 years ago, John was older than Marie.

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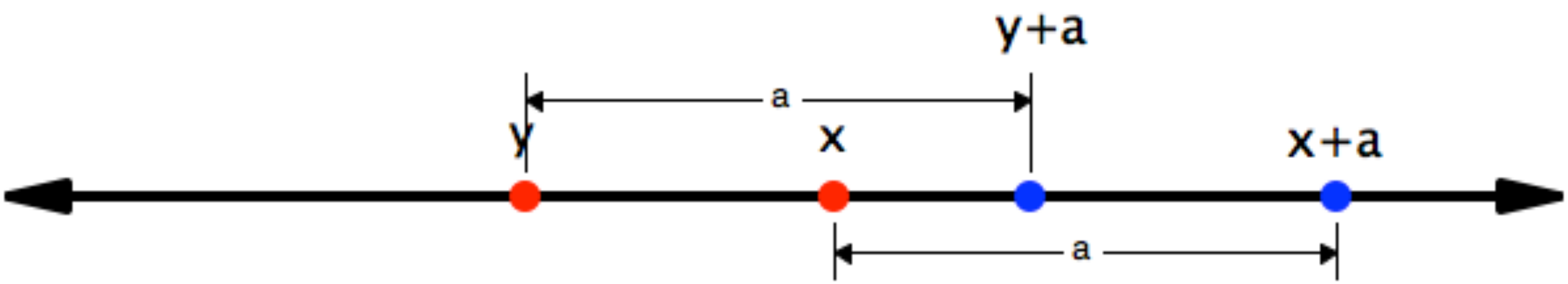
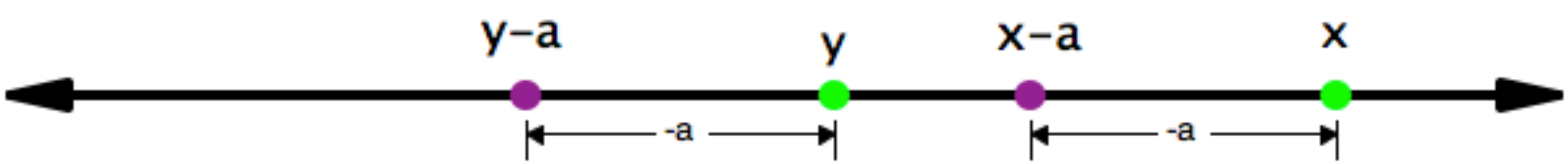
If x > y, then which is greater? x + a or y + a





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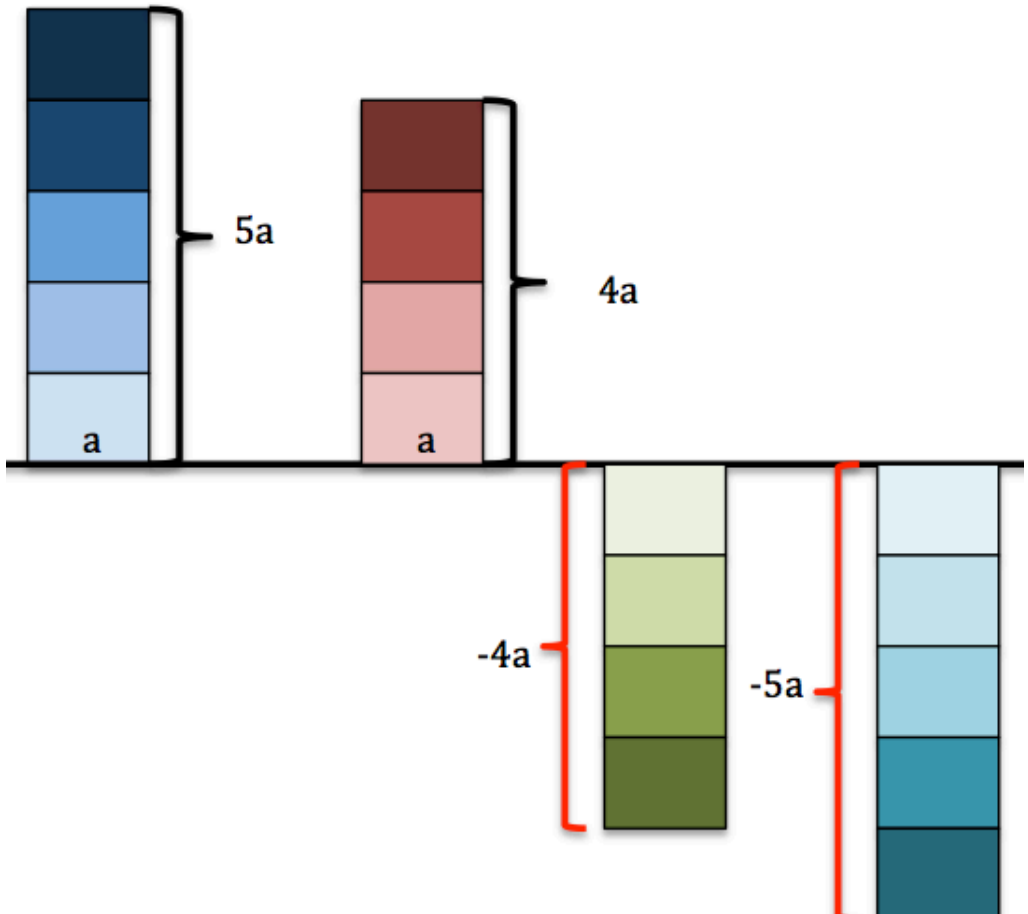
I know that 5 > 4. Which is greater? 5a or 4a



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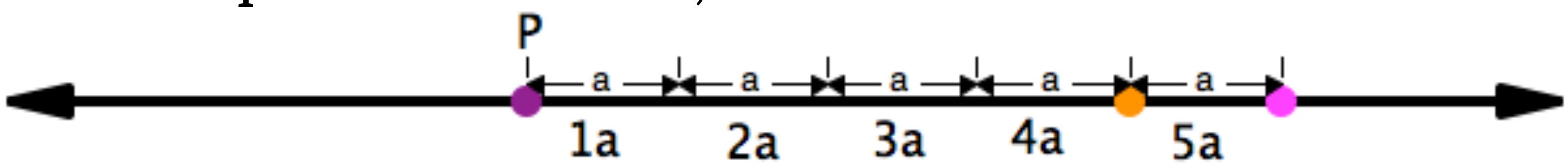
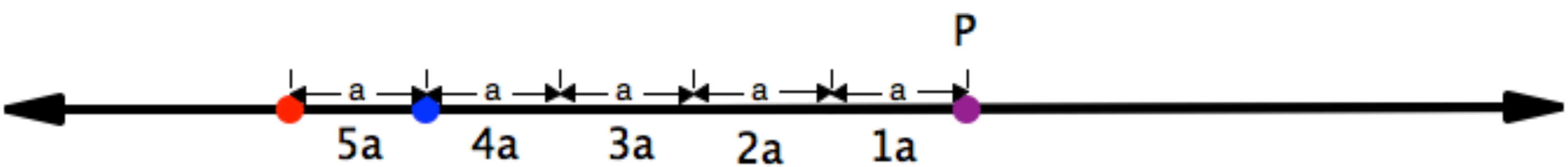
Problem 4
I know that 5 > 4. Which is greater? 5a or 4a

If “**a**” is a positive number, 5a > 4a because



If a is a negative number, 5a < 4a because



If a = 0, then 5a = 4a because 0 = 0.

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I purposely selected problems that are traditionally taught in a very procedural way.

What are the opportunities for conceptual understanding that occur as a result of encouraging multiple representations?

How can you help students make important connections between the representations?

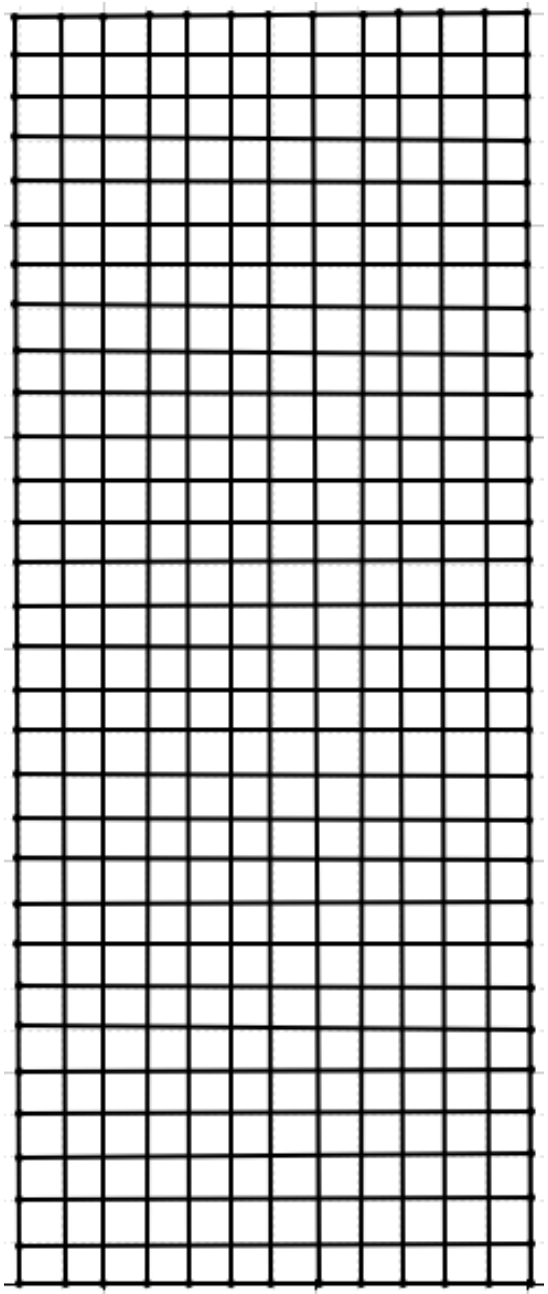
"The depth of understanding is related to the strength of connections among mathematical representations that students have internalized."

(Pape and Tchoshanov 2001; Webb, Boswinkle, and Dekker 2008)

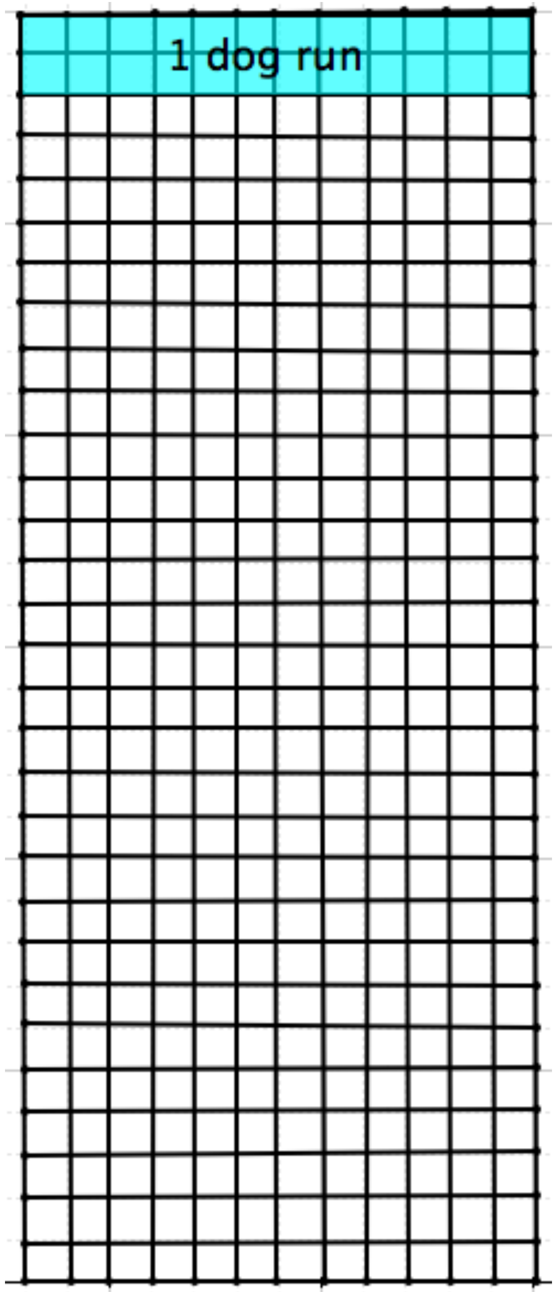
Problem 5
5.1 Pet Sitters

Carlos and Clarita are making a list of some of the issues they need to consider as part of their business plan to care for cats and dogs while their owners are on vacation.

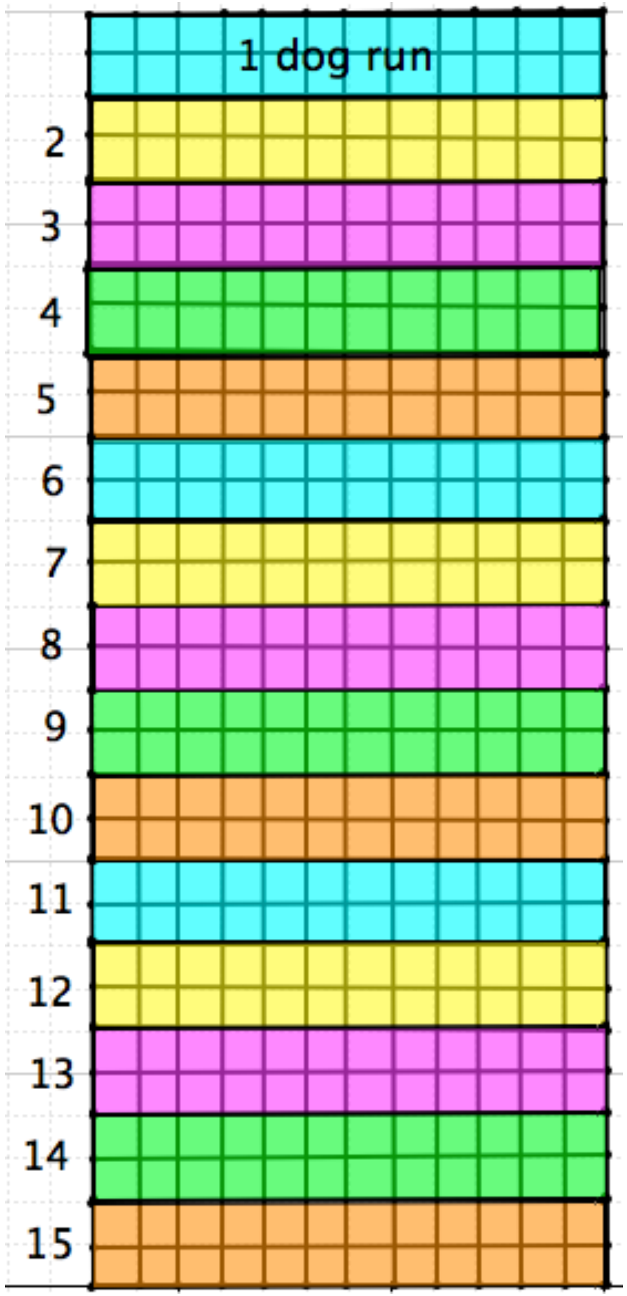
- ***Space***: Cat pens will require **6 ft^2** of space, while dog runs require **24 ft^2**. Carlos and Clarita have up to **360 ft^2** available in the storage shed for pens and runs, while still leaving enough room to move around the cages.
- ***Start-up Costs***: Carlos and Clarita plan to invest much of the **$1280** they earned from their last business venture to purchase cat pens and dog runs. It will cost **$32** for each cat pen and **$80** for each dog run.
- ***Revenue:*** They plan to charge **$8** per day for boarding each cat and **$20** per day for each dog. Their dad has suggested they plan on 12 cats and 12 dogs to begin with.



I made a grid that was 30 by 12 squares representing the 360 square feet that were available.

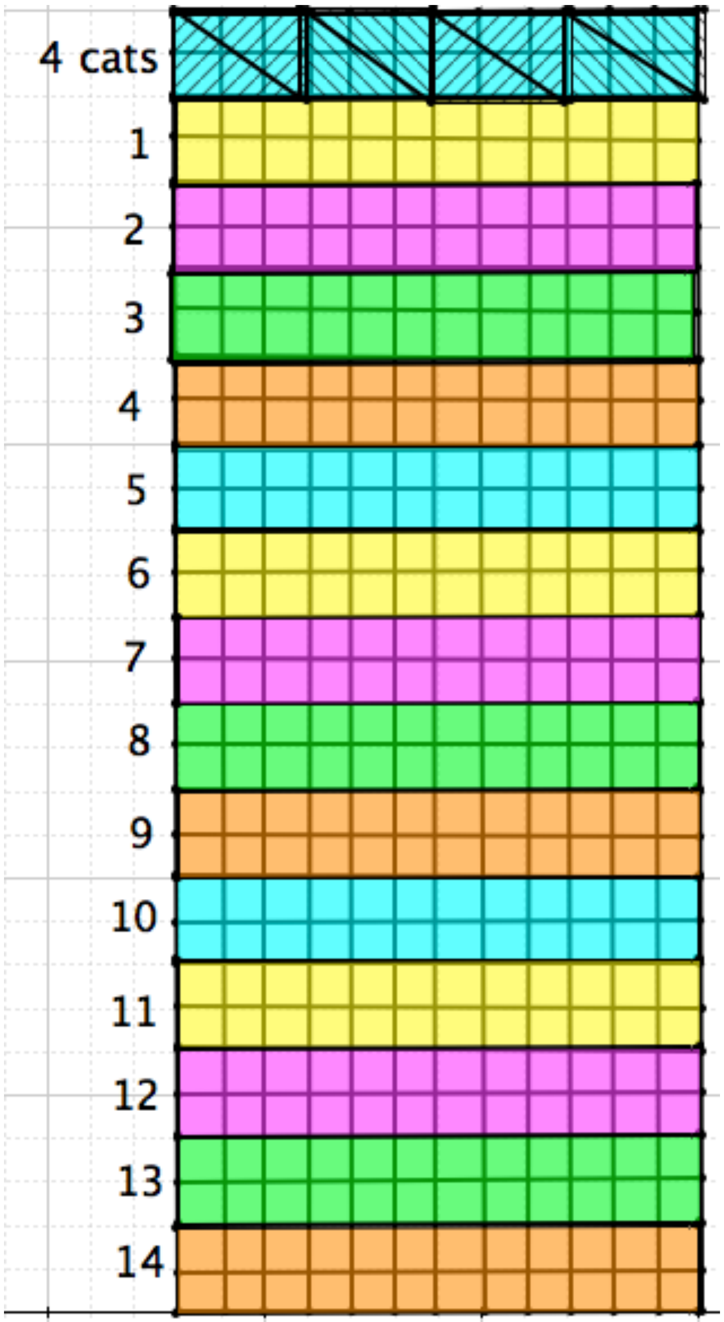


2 rows of 12 spaces represent the 24 square feet for 1 dog run.



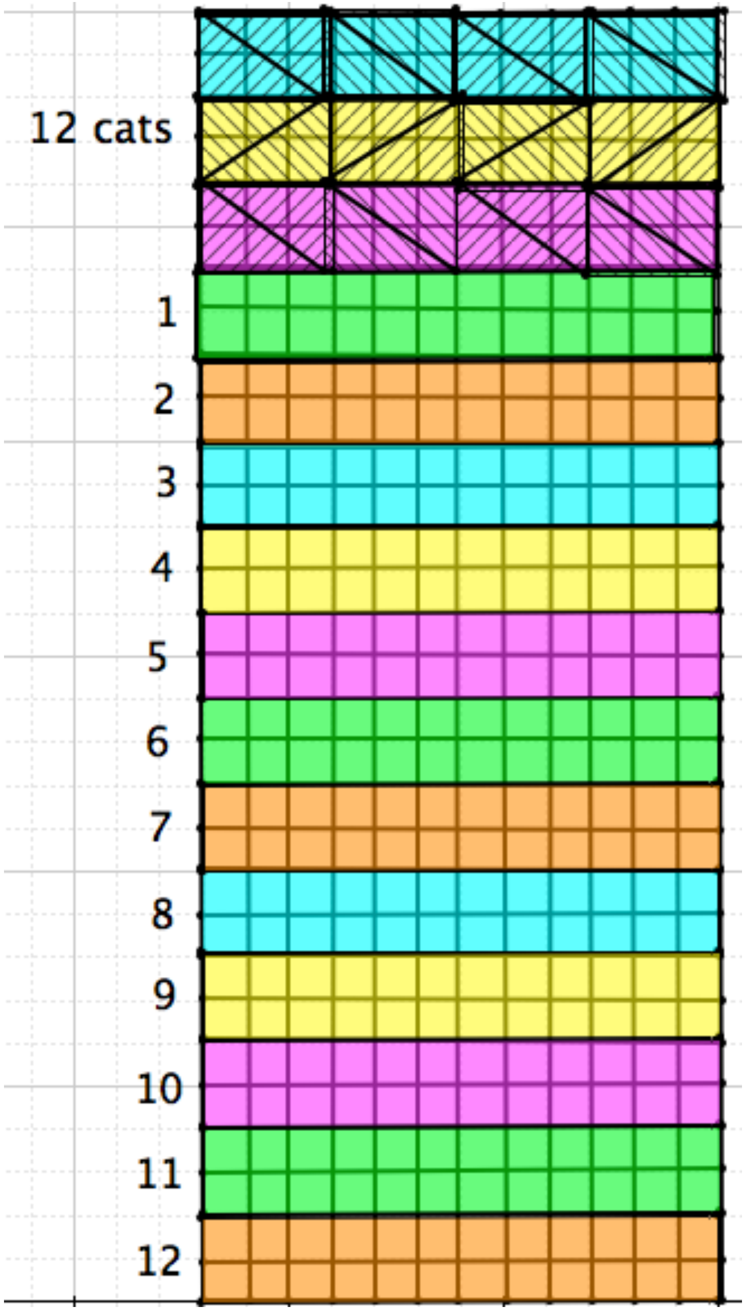
I can fit exactly 15 dog runs on the grid to fill the 360 square feet.

That would mean no cats.



Each time I eliminate 1 dog run, I can add in 4 cat pens.

So 4 cat pens and 14 dog runs work exactly in the space.



I have exactly enough space for Dad's suggestion of 12 cat pens and 12 dog pens.

I also have enough space for 60 cat pens, if I have no dogs.

CATS	DOGS
0	15
4	14
8	13
12	12
16	11
20	10
24	9
28	8
32	7
36	6
40	5
44	4
48	3
52	2
56	1
60	0

Can you see the same relationships about space in a table?

CATS	DOGS
0	**15**
4	14
8	13
12	**12**
16	11
20	10
24	9
28	8
32	7
36	6
40	5
44	4
48	3
52	2
56	1
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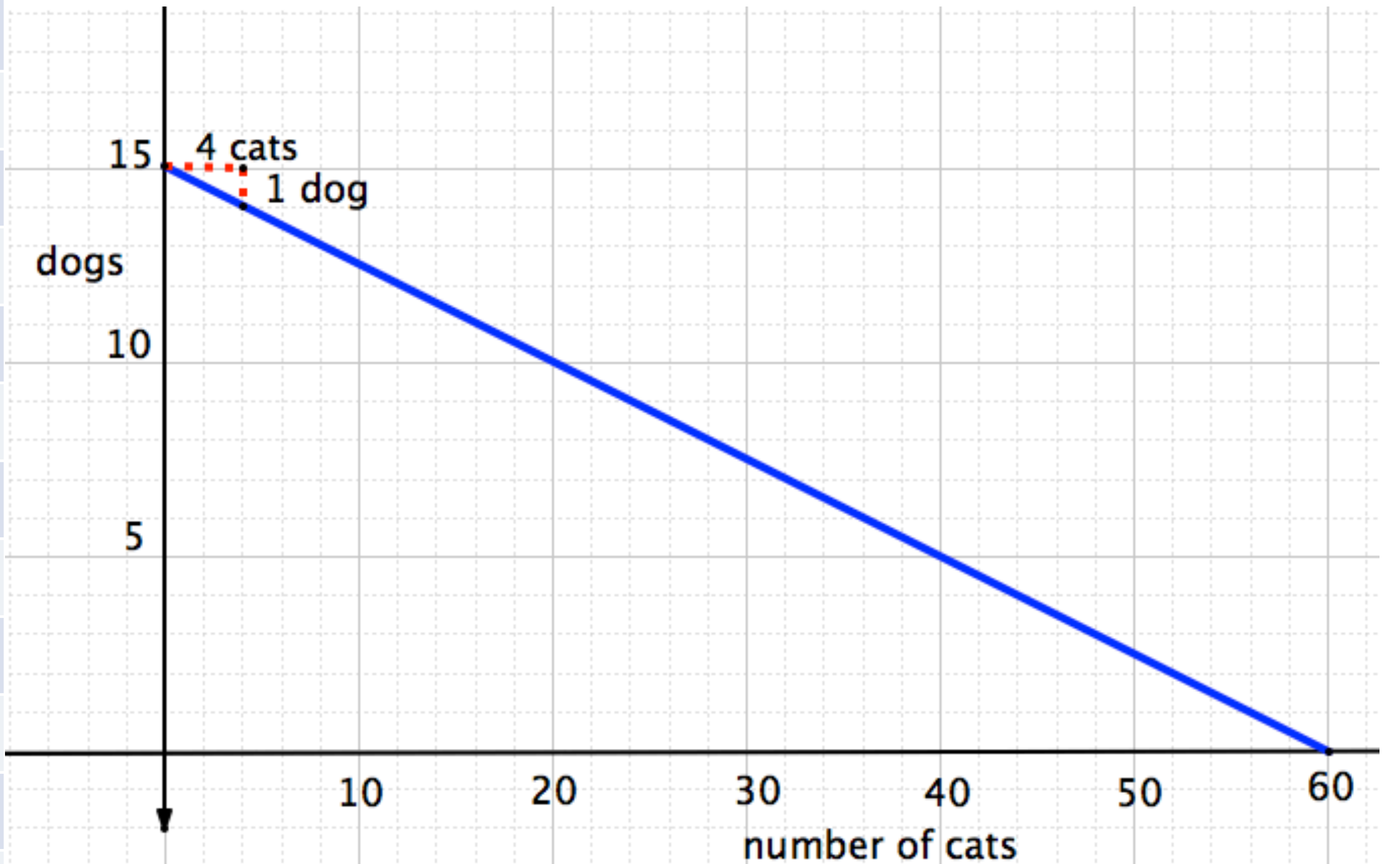
32	7
36	6
40	5
44	4
48	3
52	2
56	1
60	0

Are there ordered pairs in the table that would help you make a graph?



CATS	DOGS
0	**15**
4	14
8	13
12	**12**
16	11
20	10
24	9
28	8
32	7
36	6
40	5
44	4
48	3
52	2
56	1
60	**0**

Are there ordered pairs in the table that would help you write an equation?

- Cats require 6 ft^2 and Dogs 24 ft^2
- I know if I have 0 cats and 15 dogs the 360 ft^2 is filled.
- I know if I have 60 cats and 0 dogs the 360 ft^2 is filled.
- **The equation would be**

  $$6c + 24d = 360$$

What would the slope mean if someone wrote his equation in slope intercept form?

- D = -1/4 C + 15   for space

(Every time I eliminate a dog, I can add in 4 cats)

(If I have no cats, I have enough space for 15 dogs)

- D = -2/5 C + 16   for costs

(Every time I eliminate 2 dog runs, I can afford 5 cat pens.)

(If I don't buy any cat pens, I can afford 16 dog runs.)

But I only have space for 15 dogs!

CATS	DOGS	COST
0	15	15 X 80 = $1200
4	14	4 X 32 + 14 X 80 = $1248
8	13	8 X 32 + 13 X 80 = $1296
12	12	12 X 32 + 12 X 80 = 1344
16	11	16 X 32 + 11 X 80 = 1392
20	10	20 X 32 + 10 X 80 = 1440
24	9	24 X 32 + 9 X 80 = 1488
28	8	28 X 32 + 8 X 80 = 1536
32	7	
36	6	
40	5	
44	4	
48	3	
52	2	
56	1	
60	0	

How could a table help us think about the cost constraint?

- They have $1280 to invest
- Dog runs cost $80
- Cat pens cost $32

Are there really only 2 possibilities because of the cost?

cats

square ft. for those pets

dogs

7		174	180	186	192	198	204	210	216
6		150	156	162	168	174	180	186	19[illegible]
5		126	132	138	144	150	156	162	168
4	96	102	108	114	120	126	132	138	144
3	72	78	84	90	96	102	108	114	120
2	48	54	60	66	72	78	84	90	96
1	24	30	36	34	50	56	62	68	74
		6	12	18	24	30	36	42	48
		1	2	3	4	5	6	7	8

cats

Depth of understanding is related to the strength of connections among mathematical representations that students have internalized.

(Pape and Tchoshanov 2001; Webb, Boswinkle, and Dekker 2008)

When do students get those opportunities in math class to make connections between their representations?

Purpose: As students work with the context of making recommendations for how many dogs and cats Carlos and Clarita should plan to accommodate, they will surface many ideas, strategies and representations related to solving systems of equations and inequalities. For example, they will explore the notion of constraints since in this task the number of each type of pet that can be accommodated is limited by space and money, but many different combinations of dogs and cats are possible.

Learning mathematics is not about memorizing content or procedures but exploring ideas.

Using these "different representations is like examining the concept through a variety of lenses, with each lens providing a different perspective that make the picture (concept) richer and deeper." *Tripathi (2008) (p.439)*

Representations should become part of the fabric of the classroom's culture for reasoning and sense-making.

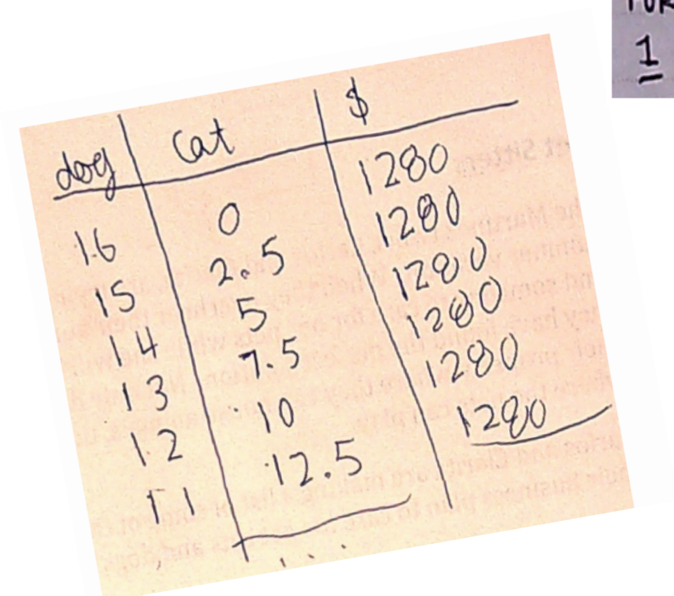
You can leverage the use of representations on a regular basis by prompting students to create them and then highlighting their value during classroom discussions.

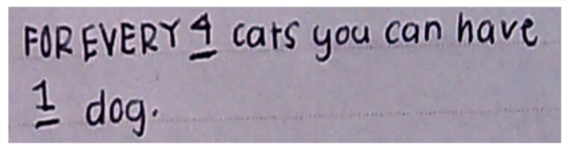
"Mathematically proficient students make sense of problems and persevere in solving them . . ." through finding access points. Often these access point are the representations that they create.

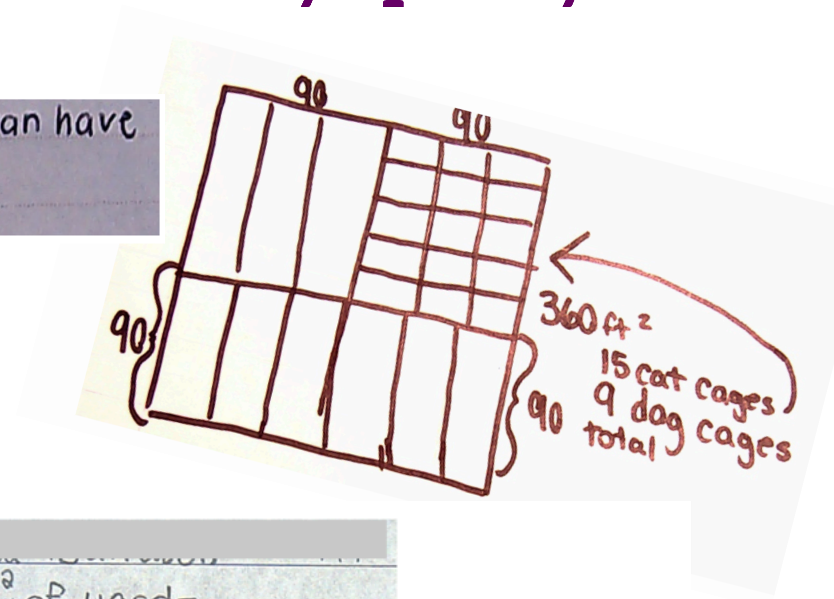
Through the construction and sharing of representations, students mentally engage with the content through offering their ideas, explanations, justifications, interpretations, reasons, evidence, perspectives, alternatives, and questions.

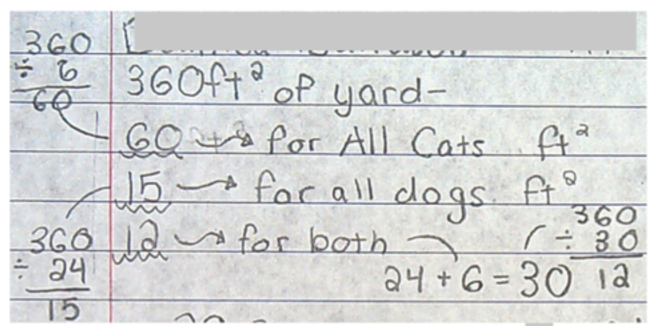
Grappling with problems can be messy.

Student representations are not always pretty.









But when we encourage our students to create them we are supporting them in productive struggle.

But we are also empowering them by giving them the tools they need for reasoning and sense-making.

Remember that:

Representations are the vehicles through which students explore content.

Connecting representations moves students' intuitive thinking towards conventional and efficient mathematics.

Learning is a consequence of **thinking**.

Thinking is a consequence of **grappling with a problem.**

Grappling with a problem is a consequence of **trying to make sense of problems and persevere in solving them**.

Trying to make sense of problems and persevere in solving them is a consequence of **creating and connecting representations.**

Learning mathematics is a consequence of **creating and connecting representations.**

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#35 Multiple Representations and Perseverance

A Tool for When the Going Gets Tough

Presenter: Janet Sutorius

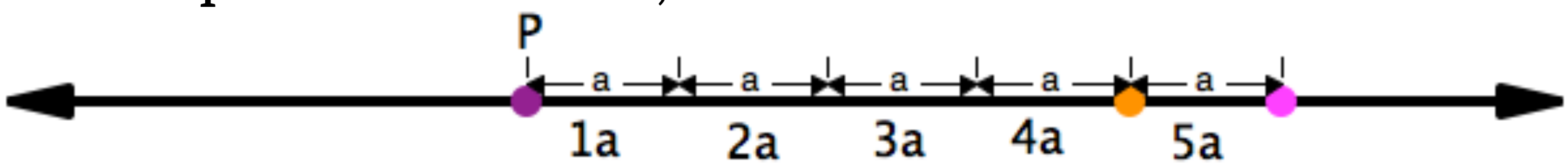
One strategy for weaving representations into the fabric of the classroom's culture is by using sentence frames to invite students to consider the value of the different types of representations.

Sentence frames invite students to think about the connections between representations. They also support students in finding the language to talk about the mathematics.

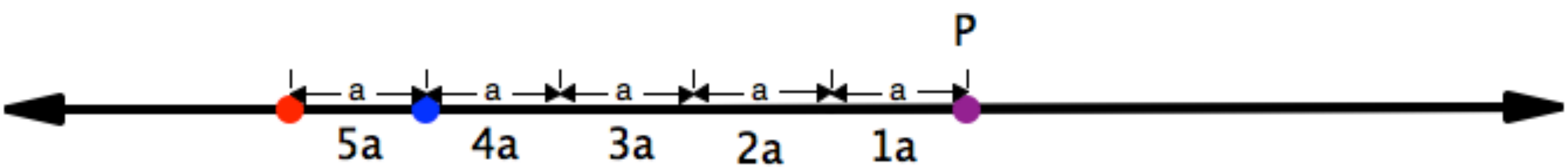
- I chose to represent ______________ using a ______________ because it was helpful to see ____
- The connection I see between the ____________________ (representation) and the _____________(representation) is ____
- In this situation, a ___________________ might be a more helpful tool than a __________________________ because ____

I know that 5 > 4. Which is greater? 5x or 4x

If x is a positive number, 5x > 4x.



If x is a negative number, 5x < 4x



If x = 0, then 5x = 4x.

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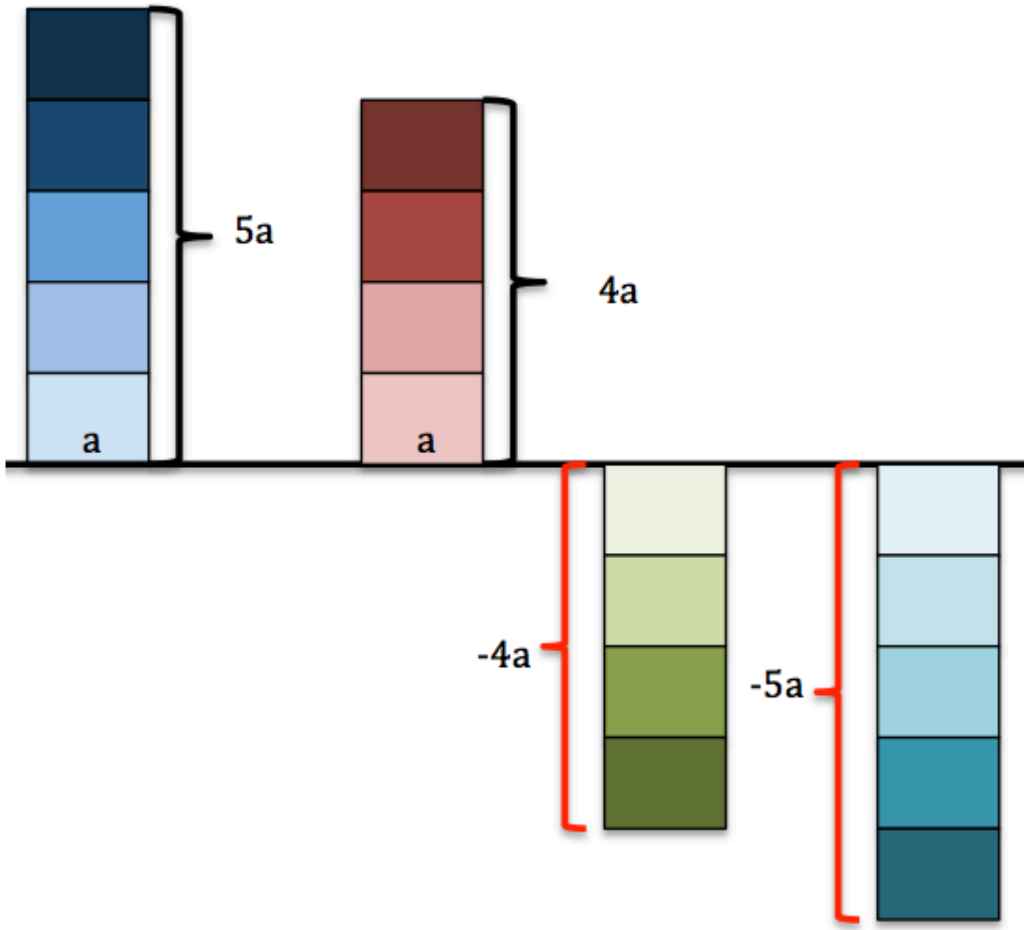
I know that 5 > 4. Which is greater? 5x or 4x



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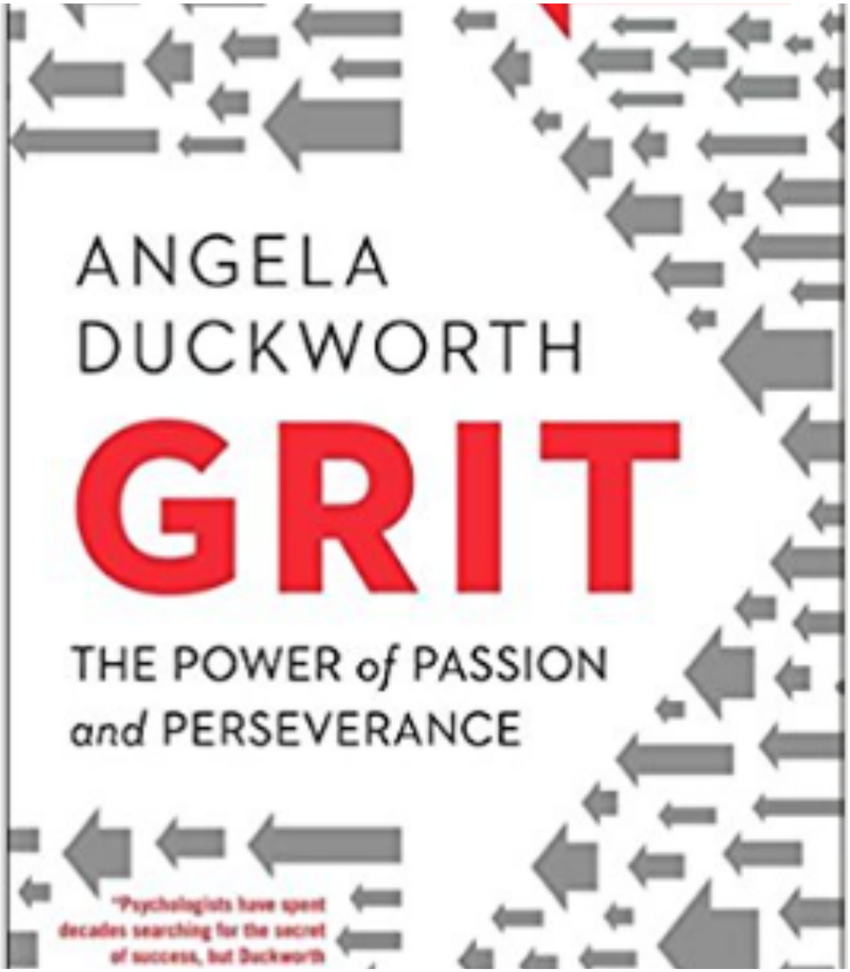
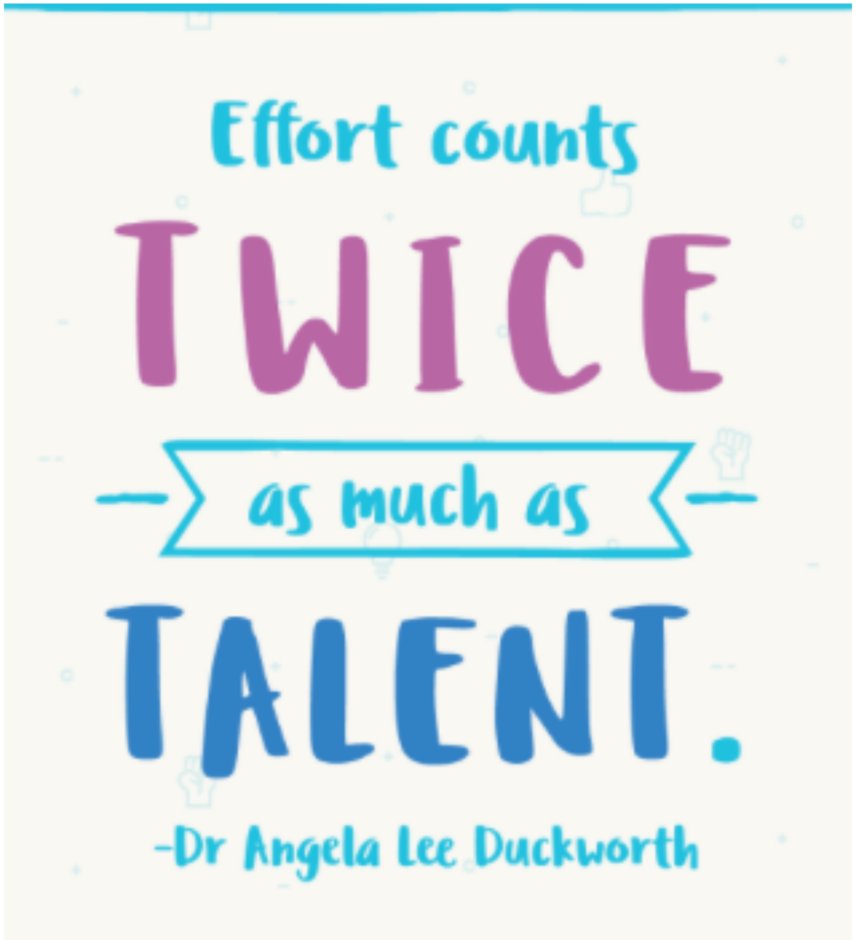
What Is Grit?



- Tenacity
- Fortitude
- Endurance
- Drive
- Stamina
- Resilience
- Hardiness
- Persistence
- Sustainability

Who has Grit?



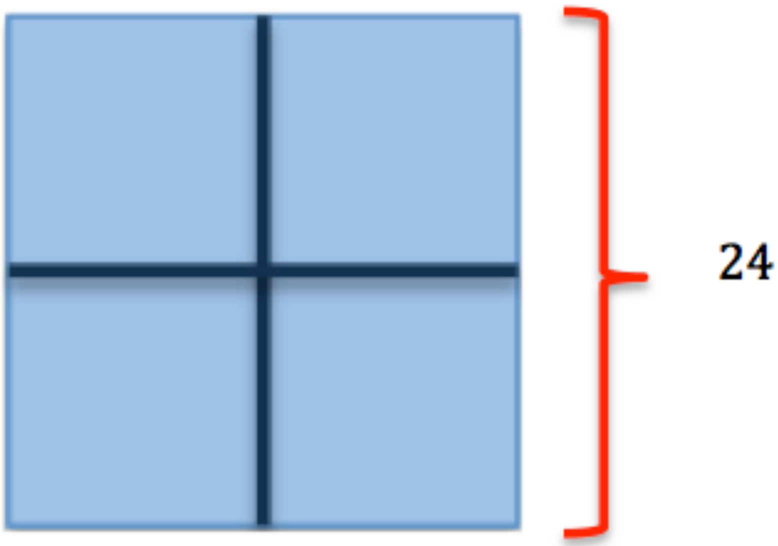
- Tenacity
- Fortitude
- Endurance
- Drive
- Stamina
- Resilience
- Hardiness
- Persistence
- Sustainability

Problem 2
4.1 Cafeteria Actions and Reactions

Elvira, the cafeteria manager, has just received a shipment of new trays with the school logo prominently displayed in the middle of the tray. After unloading 4 cartons of trays in the pizza line, she realizes that students are arriving for lunch and she will have to wait until lunch is over before unloading the remaining cartons. The new trays are very popular and in just a couple of minutes 24 students have passed through the pizza line and are showing off the school logo on the trays. At this time, Elvira decides to divide the remaining trays in the pizza line into 3 equal groups so she can also place some in the salad line and the sandwich line, hoping to attract students to the other lines. After doing so, she realizes that each of the three serving lines has only 12 of the new trays.

- "That's not many trays for each line. I wonder how many trays there were in each of the cartons I unloaded?"

4.1 Cafeteria Actions and Reactions



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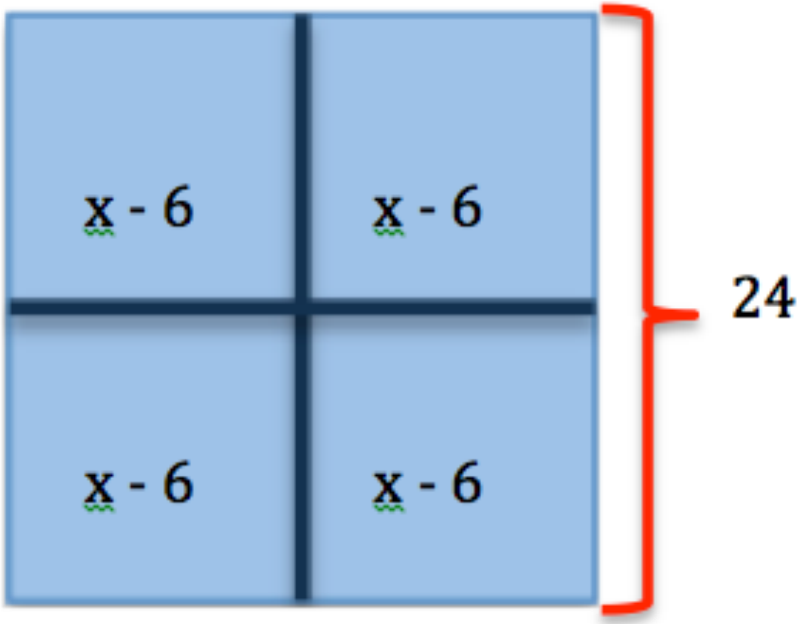

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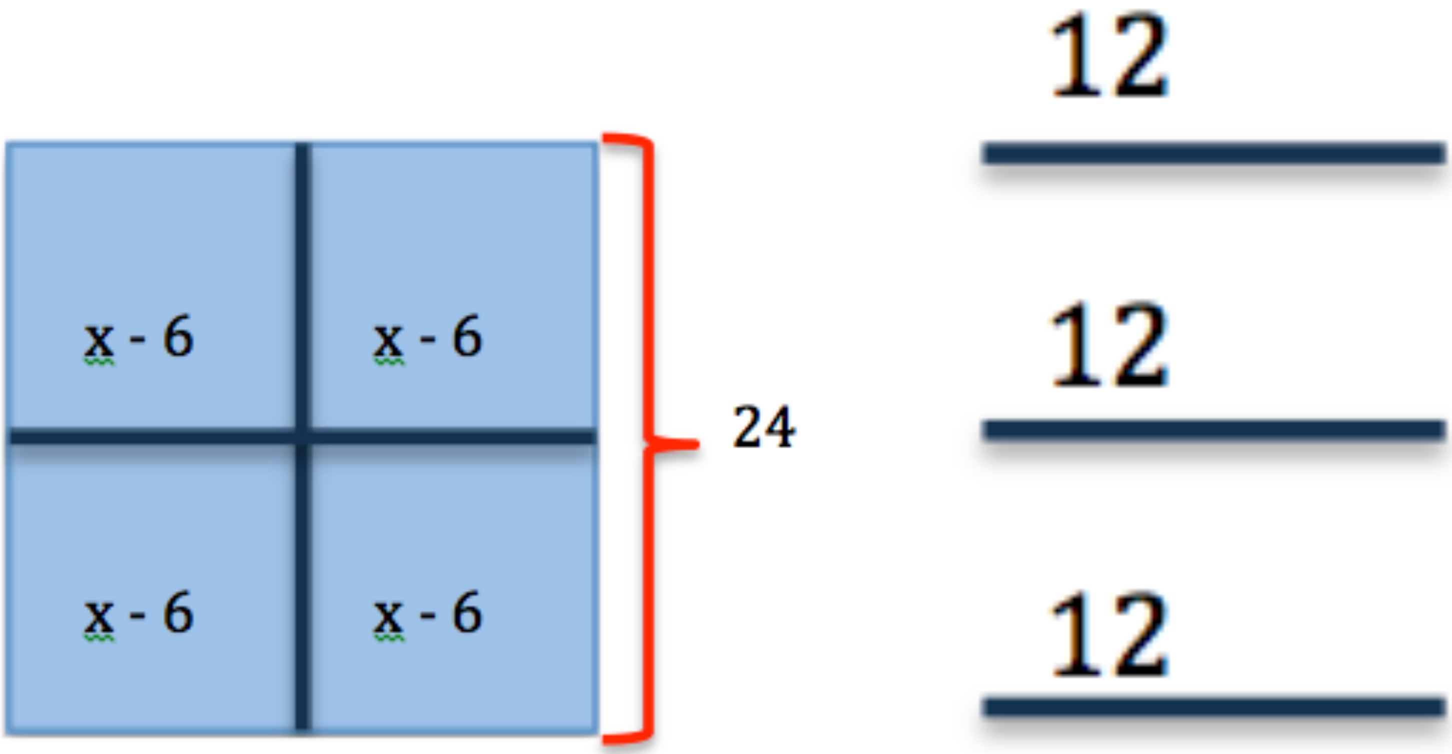

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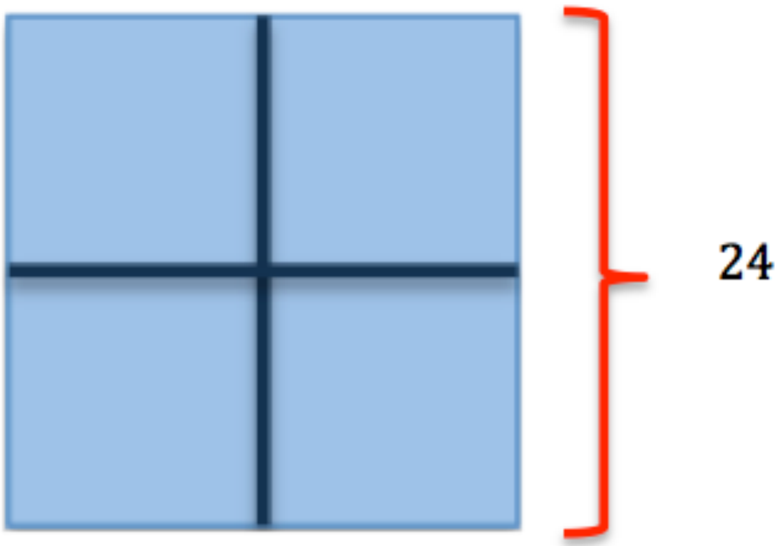

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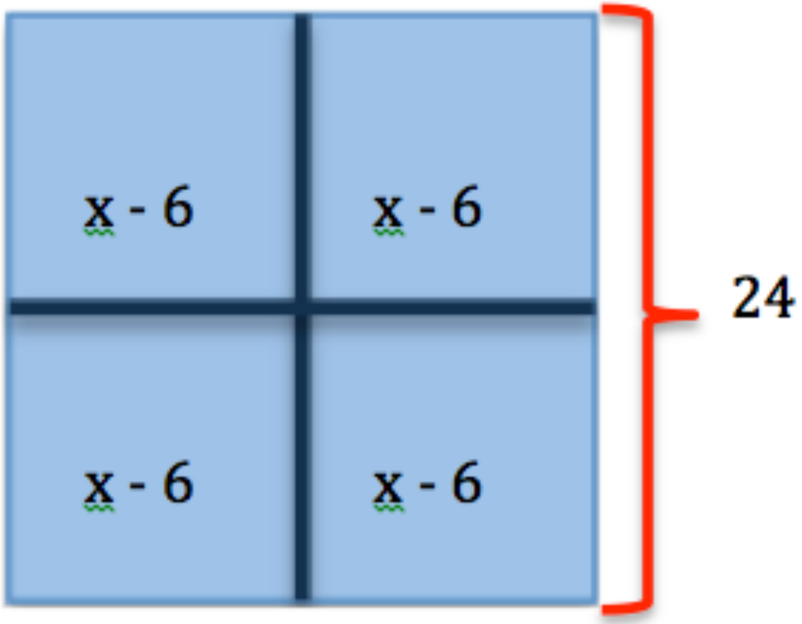

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