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Transforming Mathematics Education

SECONDARY MATH THREE

An Integrated Approach

Standard Teacher Notes

MODULE 7

Trigonometric Functions, Equations & Identities

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The Mathematics Vision Project

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7.1 High Noon and Sunset Shadows

A Develop Understanding Task

In this task we revisit the amusement park Ferris wheel that caused Carlos so much anxiety. Recall the following facts from previous tasks:

- The Ferris wheel has a radius of 25 feet
- The center of the Ferris wheel is 30 feet above the ground
- The Ferris wheel makes one complete rotation counterclockwise every 20 seconds

The amusement park Ferris wheel is located next to a high-rise office complex. At sunset, the moving carts cast a shadow on the exterior wall of the high-rise building. As the Ferris wheel turns, you can watch the shadow of a rider rise and fall along the surface of the building. In fact, you know an equation that would describe the motion of this "sunset shadow."

1. Write the equation of this "sunset shadow."

At noon, when the sun is directly overhead, a rider casts a shadow that moves left and right along the ground as the Ferris wheel turns. In fact, you know an equation that would describe the motion of this "high noon shadow."

2. Write the equation of this "high noon shadow."

3. Based on your previous work, you probably wrote these equations in terms of the angle of rotation being measured in degrees. Revise you equations so the angle of rotation is measured in radians.

   a. The “sunset shadow” equation in terms of radians:

   b. The “high noon shadow” equation in terms of radians:

4. In the equations you wrote in question 3, where on the Ferris wheel was the rider located at time $t = 0$? (We will refer to the position as the rider’s *initial position* on the wheel.)

5. Revise your equations from question 3 so that the rider’s initial position at $t = 0$ is at the top of the wheel.

   a. The “sunset shadow” equation, initial position at the top of the wheel:

   b. The “high noon shadow” equation, initial position at the top of the wheel:

6. Revise your equations from question 3 so that the rider’s initial position at $t = 0$ is at the bottom of the wheel.

   a. The “sunset shadow” equation, initial position at the bottom of the wheel:

   b. The “high noon shadow” equation, initial position at the bottom of the wheel:

7. Revise your equations from question 3 so that the rider's initial position at $t = 0$ is at the point farthest to the left of the wheel.

    a. The "sunset shadow" equation, initial position at the point farthest to the left of the wheel:

    b. The "high noon shadow" equation, initial position at the point farthest to the left of the wheel:

8. Revise your equations from question 3 so that the rider's initial position at $t = 0$ is halfway between the farthest point to the right on the wheel and the top of the wheel.

    a. The "sunset shadow" equation, initial position halfway between the farthest point to the right on the wheel and the top of the wheel:

    b. The "high noon shadow" equation, initial position halfway between the farthest point to the right on the wheel and the top of the wheel:

9. Revise your equations from question 3 so that the wheel rotates twice as fast.

    a. The "sunset shadow" equation for the wheel rotating twice as fast:

    b. The "high noon shadow" equation for the wheel rotating twice as fast:

10. Revise your equations from question 3 so that the radius of the wheel is twice as large and the center of the wheel is twice as high.

    a. The "sunset shadow" equation for a radius twice as large and the center twice as high:

    b. The "high noon shadow" equation for a radius twice as large and the center twice as high:

11. Carlos wrote his "sunset equation" for the height of the rider in question #5 as $h(t) = 50\sin\left(\frac{\pi}{10}t + \frac{\pi}{2}\right) + 30$. Clarita wrote her equation for the same problem as $h(t) = 50\sin\left(\frac{\pi}{10}(t+5)\right) + 30$.

    a. Are both of these equations equivalent? How do you know?

    b. Carlos says his equation represents starting the rider at an initial position at the top of the wheel. What does Clarita's equation represent?

7.1 High Noon and Sunset Shadows – Teacher Notes

A Develop Understanding Task

Purpose: The purpose of this task is to develop strategies for transforming the Ferris wheel functions so that the function and graphs represent different initial starting positions for the rider. Students have already considered vertical translations by moving the center of the Ferris wheel up or down, resulting in the midline of the graph being translated vertically. They have also considered horizontal and vertical dilations of the graph by changing the radius of the Ferris wheel or the speed of rotation, resulting in varying the amplitude or the period of the graph. In this task, horizontal translations of the graph are considered. Students may also note that sine and cosine graphs are interchangeable, as long as the graph is shifted horizontally by an appropriate amount.

Core Standards Focus:

F.TF.5 Choose trigonometric functions to model periodic phenomena with specified amplitude, frequency, and midline.★

F.BF.3 Identify the effect on the graph of replacing f(x) by f(x) + k, k f(x), f(kx), and f(x + k) for specific values of k (both positive and negative); find the value of k given the graphs. Experiment with cases and illustrate an explanation of the effects on the graph using technology. Include recognizing even and odd functions from their graphs and algebraic expressions for them.

Related Standards: F.BF.1c

Standards for Mathematical Practice:

SMP 4 – Model with mathematics

SMP 5 – Use appropriate tools strategically

Vocabulary: This task varies the *initial position* of the rider on the Ferris wheel.

The Teaching Cycle:

Launch (Whole Class):

Model the scenario given in this task by using a strong flashlight to represent the sun. Have a student hold a pencil so it is parallel to the ground and move it in a continual circular path to model the rider on the Ferris wheel. Note the shadow of the pencil as it moves up and down the wall when the flashlight is pointed horizontally towards the pencil (the "sunset shadow") or the shadow of the pencil as it moves back and forth across the ground when the flashlight is held above the pencil and pointed towards the ground (the "high noon shadow").

After modeling the context, assign students to answer questions 1-3 on the first page of the task. Remind students that they have written equations for the vertical and horizontal motion of the rider on this Ferris wheel in terms of degrees on previous tasks (see 6.2 and 6.5). Question 3 asks students to revise these equations in terms of radians. (Note that we will use *y(t)* to represent the height of the rider—the motion of the sunset shadow—and *x(t)* to represent the horizontal position of the rider—the motion of the high noon shadow. You may want to suggest this notation to your students, also.

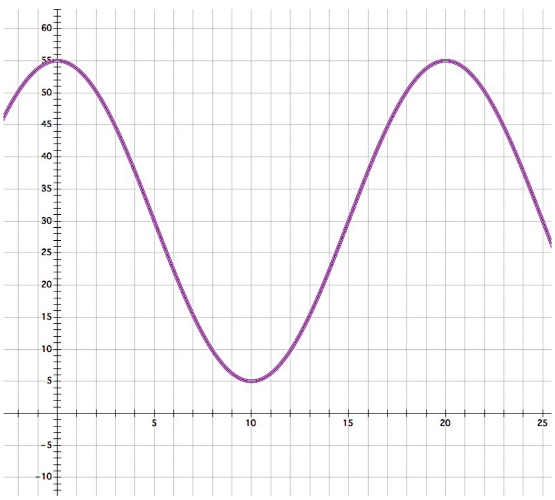
$$y(t) = 25\sin\left(\frac{\pi}{10}t\right) + 30$$
$$x(t) = 25\cos\left(\frac{\pi}{10}t\right)$$

Once students have agreed on the equations for the motion of the high noon and sunset shadows, and can explain why $b = \frac{\pi}{10}$ in the expression $a\sin(bt) + d$, assign them to work on the remainder of the task where the *initial* or *starting position* of the rider will be located at different points around the wheel.

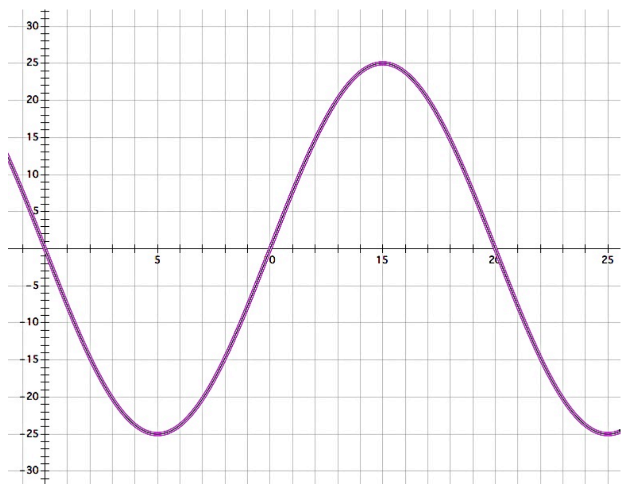
Explore (Small Group):

If students are having difficulty getting started on this work, suggest that they sketch the graphs of the motion of the high noon and sunset shadows such that at time $t = 0$ the rider is at the initial position given in the scenario. This graphical representation of the rider should help them think about how they might modify their equations from question 3 to produce these new graphs. Students might use technology to test out their proposed equations to see if they match the graphs they have sketched by hand. For example, for question 5 (starting position of the rider at the top of the wheel), students should sketch the following graphs:

Sunset Shadow



High Noon Shadow



Students should notice that the graph representing the height of the rider—the motion of the sunset shadow—looks like a cosine graph, and that the graph representing the horizontal position of the rider—the motion of the high noon shadow—looks like a sine graph that has been reflected over the x-axis. This should lead to the following proposals for the functions:

$$y(t) = 25\cos\left(\frac{\pi}{10}t\right) + 30$$

$$x(t) = -25\sin\left(\frac{\pi}{10}t\right)$$

Accept these as correct equations, and then challenge students to try to find alternative equations so that we can continue to use a sine function to represent the vertical motion of the rider and a cosine function to represent the horizontal motion. Suggest that they might do so by modifying the expression inside the function, thereby translating the sine or cosine function horizontally. Let students guess and check different values for c in the expression $a\sin(bt + c) + d$ if that is their preferred strategy, but ultimately students should be able to explain why their choice for c makes sense. Press for explanations such as, "At time t = 0 we want the function to use the value of the

$\sin\frac{\pi}{2}$, so we will start the rider at the highest position on the graph."

It may be helpful to hold a short discussion regarding students' results on question 5 before having students work on the remainder of the task. Students should be able to explain why the following equations also work for the graphs in question 5:

$$y(t) = 25\sin\left(\frac{\pi}{10}t + \frac{\pi}{2}\right) + 30$$
$$x(t) = 25\cos\left(\frac{\pi}{10}t + \frac{\pi}{2}\right)$$

Ask students to continue to find ways to use a sine function to represent the vertical motion of the rider and a cosine function to represent the horizontal motion of the rider as they work on the remainder of the problems in the task. This will keep the task focused on the horizontal transformations of the graphs. If some students finish the task more quickly than others, challenge them to go back and write other equations for each situation that might involve a different trigonometric function or a different horizontal transformation.

Discuss (Whole Class):

Post these generic forms for describing the motion of the high noon and sunset shadows:

The sunset shadow: $y(t) = a\sin(bt + c) + d$

The high noon shadow: $x(t) = a\cos(bt + c) + d$

Point out to students that the parameter c translates the sine or cosine graph horizontally and is sometimes referred to as a *phase shift* in scientific applications.

Select students who can explain their choices of values for c to present problems 6-8. After an equation has been presented and explained, it may be helpful to have other students calculate a few specific values of the function, such as at $t = 0$, $t = 5$, $t = 10$ or $t = 15$ seconds, to verify that the function works at these specific instances in time.

Given time, you might want to have students present some of the alternative equations they found to describe these graphs, pointing out that sine and cosine functions can be used interchangeably, as long as appropriate phase shifts are considered.

Aligned Ready, Set, Go: Trigonometric Functions, Equations and Identities 7.1

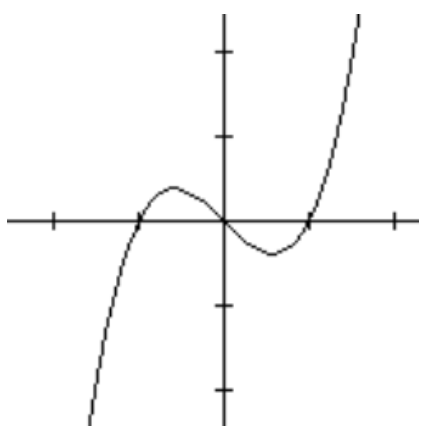
READY, SET, GO! Name Period Date

READY

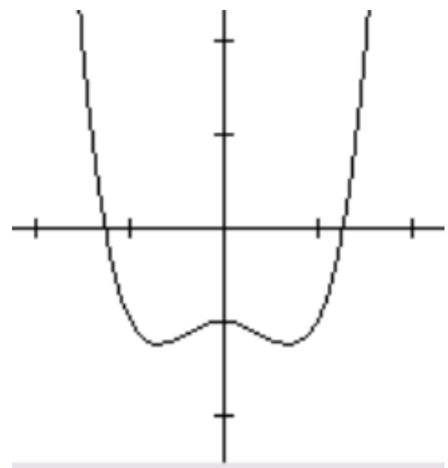
Topic: Recalling invertible functions and even and odd functions

Indicate which of the following functions have an inverse that is a function. If the function has an inverse, sketch it in. (Remember, the inverse will reflect across the y = x line. Sketch that in, too.) Finally, label each one as *even, odd*, or *neither*. Recall that an *even* function is symmetric with the y-axis, while an *odd* function is symmetric with respect to the origin.

1,



2.



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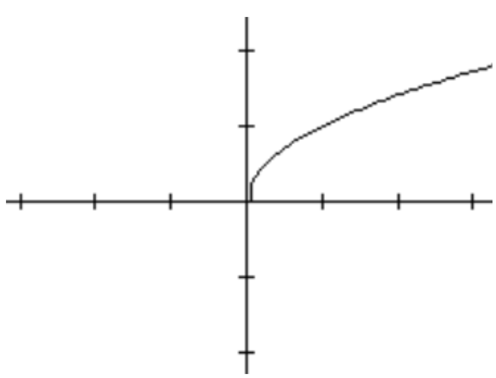

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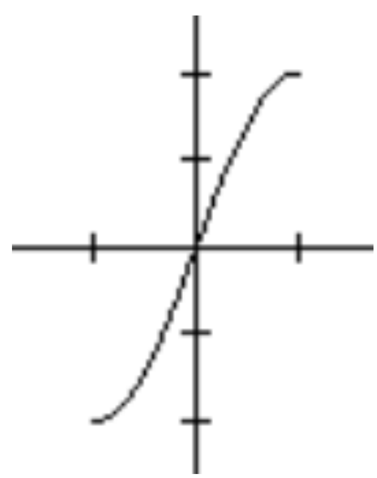
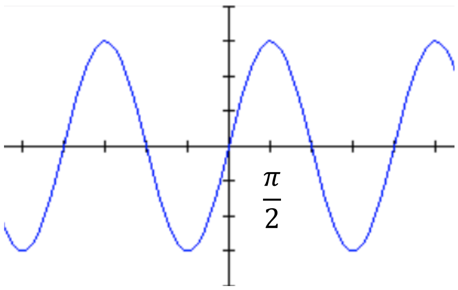


SET

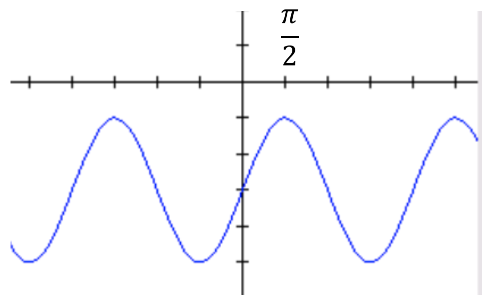
Topic: Connecting transformed trig graphs with their equations

State the period, amplitude, vertical shift, and phase shift of the function shown in the graph. Then write the equation. Use the same trigonometric function as the one that is given.

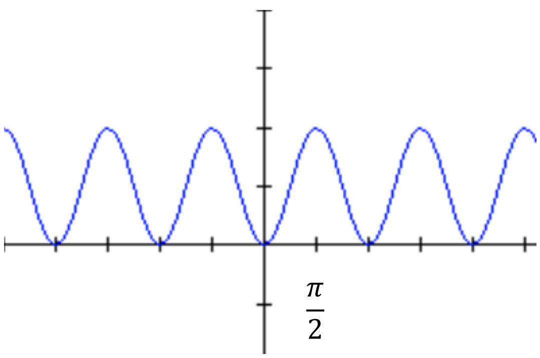
5. $y = \sin x$



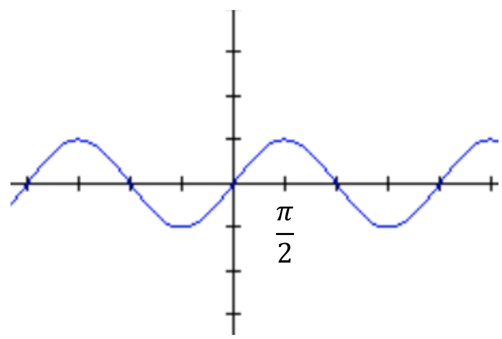
6. $y = \sin x$



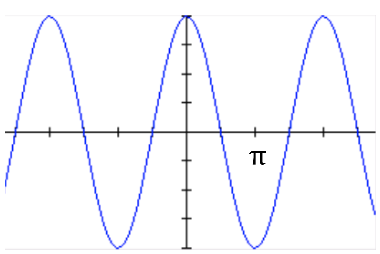
7. $y = \cos x$



8. $y = \cos x$



9. $y = \sin x$



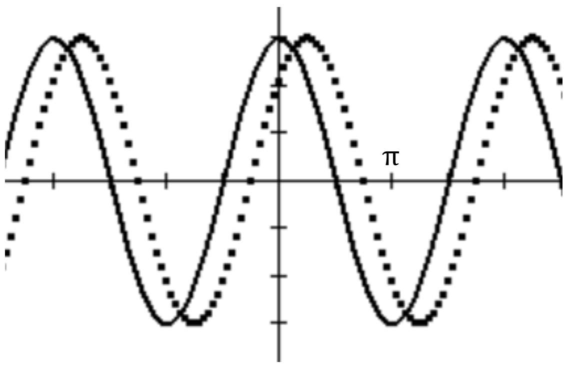
10. The cofunction identity states that $\sin\theta = \cos(90° - \theta)$ *and* $\sin(\theta - 90°) = \cos\theta$. How does this identity relate to the graph in #9?

Explain where you would see this identity in a right triangle.

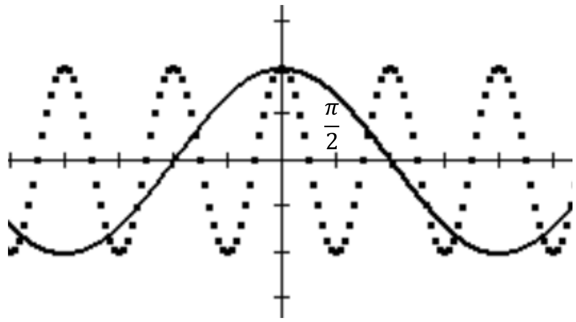
Describe the relationships between the graphs of *f (x)* - solid and *g (x)* - dotted.

Then write their equations.

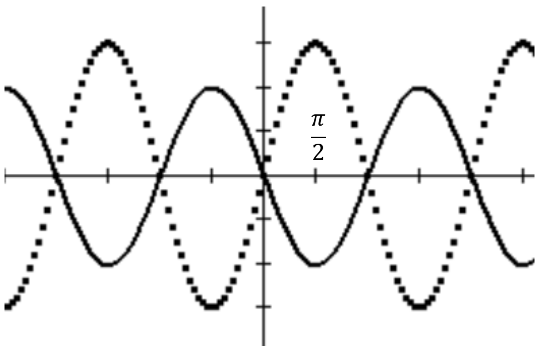
11.



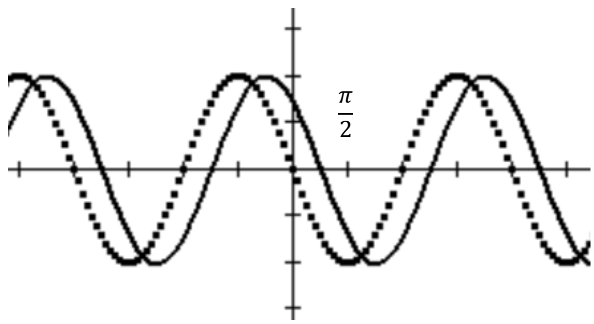
12.



13. This graph could be interpreted as a shift or a reflection. Write the equations both ways.

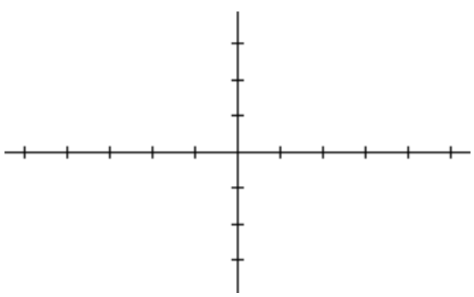


14.

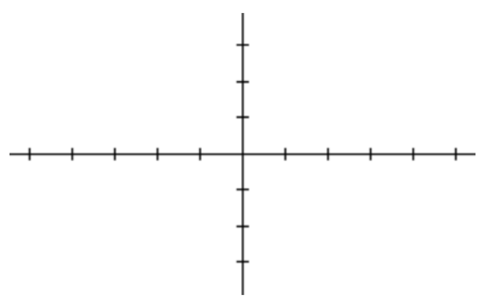


Sketch the graph of the function.
(Include 2 full periods. Label the scale of your horizontal axis.)

15. $y = 3\sin\left(x - \frac{\pi}{2}\right)$



16. $y = -2\cos(x + \pi)$



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GO

Topic: Finding angles of rotation for the same trig ratio

Name two values for θ (angles of rotation) that have the given trig ratio. $0 < \theta \leq 2\pi$.

17. $\sin\theta = \frac{\sqrt{2}}{2}$

18. $\cos\theta = \frac{\sqrt{2}}{2}$

19. $\cos\theta = -\frac{1}{2}$

20. $\sin\theta = 0$

21. $\sin\theta = -\frac{\sqrt{3}}{2}$

22. $\cos\theta = -\frac{\sqrt{3}}{2}$

23. For which angles of rotation does $\sin\theta = \cos\theta$?

7.2 High Tide

A Solidify Understanding Task

Perhaps you have built an elaborate sand castle at the beach only to have it get swept away by the in-coming tide.

Spring break is next week and you are planning another trip to the beach. This time you decide to pay attention to the tides so that you can keep track of how much time you have to build and admire your sand castle.

You have a friend who is in calculus who will be going on spring break with you. You give your friend some data from the almanac about high tides along the ocean, as well as a contour map of the beach you intend to visit, and ask her to come up with an equation for the water level on the beach on the day of your trip. According to your friend's analysis, the water level on the beach will fit this equation:

$$f(t) = 20\, sin\left(\frac{\pi}{6}t\right)$$

In this equation, $f(t)$ represents how far the waterline is above or below its average position. The distance is measured in feet, and t represents the elapsed time (in hours) since midnight.

1. What is the highest up the beach (compared to its average position) that the waterline will be during the day? (This is called *high tide*.) What is the lowest that the waterline will be during the day? (This is called *low tide*.)

2. Suppose you plan to build your castle right on the average waterline just as the water has moved below that line. How much time will you have to build your castle before the incoming tide destroys your work?

3. Suppose you want to build your castle 10 feet below the average waterline to take advantage of the damp sand. What is the maximum amount of time you will have to make your castle? How can you convince your friend that your answer is correct?

4. Suppose you want to build your castle 15 feet above the average waterline to give you more time to admire your work. What is the maximum amount of time you will have to make your castle? How can you convince your friend that your answer is correct?

5. You may have answered the previous questions using a graph of the tide function. Is there a way you could use algebra and the inverse sine function to answer these questions. If so, show your work.

    a. Algebraic work for question 3:

b. Algebraic work for question 4:

6. Suppose you decide you only need two hours to build and admire your castle. What is the lowest point on the beach where you can build it? How can you convince your friend that your answer is correct?

7.2 High Tide – Teacher Notes

A Solidify Understanding Task

Purpose: In this task students solidify their understanding of using trigonometric functions to model periodic behavior by applying trigonometry to a context that is periodic (high and low tides), but not circular motion. They learn how to interpret amplitude and period in terms of this new context. They also consider using inverse trigonometric functions to answer questions about the time at which the tide reaches various heights. Students answer these inverse questions both graphically and algebraically.

Core Standards Focus:

F.TF.5 Choose trigonometric functions to model periodic phenomena with specified amplitude, frequency, and midline.★

F.BF.4 Find inverse functions.

a. Solve an equation of the form f(x) = c for a simple function f that has an inverse and write an expression for the inverse.

c. (+) Read values of an inverse function from a graph or a table, given that the function has an inverse.

Standards for Mathematical Practice:

SMP 4 – Model with mathematics

The Teaching Cycle:

Launch (Whole Class):

Introduce the context of tides to students and have them answer questions 1 and 2 individually, and then discuss their answers as a class. Verify that students are able to interpret the parameters of this equation in terms of this new context. They should be able to explain that the 20 in the equation tells them that the tide will range 20 feet above or below average tide, and that the $^{\pi}/_{6}$

tells them that the period of the tide function (one complete cycle of high and low tides) is 12 hours, which can be found by calculating $\frac{2\pi}{\pi/6}$. If students are having difficulty relating a trigonometric function to a non-circular context, ask then how they would answer these questions if the equation was referring to a Ferris wheel. Point out that even though the height of the tides has nothing to do with angles of rotation, we can "borrow" the periodic behavior of circular trigonometric functions to describe other periodic contexts such as tides, vibrating strings, or the cycle of average temperatures over the course of a year.

Following this introduction, have students work on questions 3-5, and question 6 if they have time.

Explore (Small Group):

It is anticipated that students will answer questions 3 and 4 by referring to the graph of the situation, perhaps by plotting a horizontal line at $y = -10$ (for question 3) or $y = 15$ (for question 4) and then using the features of the calculator to determine the points of intersection of these lines with the trigonometric graph. This strategy is an important one for all students to consider as it develops conceptual understanding for the problem, so encourage and support it. Select students to present during the discuss stage of the lesson who can describe their work using this strategy. Also watch for students who use tables to answer questions 3 and 4 and select students to present this strategy also, if it is present in the work of the students. Note that since the tide goes through two complete cycles during a 24-hour day, there are actually two intervals of time when the sand castle building criteria can be satisfied for questions 3 and 4. Make sure students are answering the questions in terms of the time of the day—morning, afternoon or evening—when the sand castle could be built.

Students should be familiar with using inverse functions, and using inverse trigonometric functions to find missing angles in right triangles. They probably will not have considered the issues that arise when inverse trigonometric functions are applied to periodic functions. Allow these issues to surface naturally as students work on their algebraic approaches to answering questions 3 and 4 (see question 5). Listen for how students deal with these issues and select students to discuss these issues and how they resolved them during the whole class discussion. Referring to the graphs of the situations will help make these issues more clear. For example, students might perform the following algebra for question 3:

$$-10 = 20\sin\left(\frac{\pi}{6}t\right)$$
$$\frac{-10}{20}=\sin\left(\frac{\pi}{6}t\right)$$
$$-\frac{1}{2} = \sin\left(\frac{\pi}{6}t\right)$$
$$\sin^{-1}\left(-\frac{1}{2}\right) = \sin^{-1}\left(\sin\left(\frac{\pi}{6}t\right)\right)$$
$$\sin^{-1}\left(-\frac{1}{2}\right) = \frac{\pi}{6}t$$

At this point the question might arise, "What is the inverse sine of -1/2?" That is, "What is the angle whose sine is -1/2?" If we examine the unit circle we observe that the sine function has a value of -1/2 at the principal values of $\frac{7\pi}{6}$ and $\frac{11\pi}{6}$. Using these values to solve for t we find we can build our sand castle between 7:00 and 11:00 a.m. in the morning. However, the graph suggests another interval of time between 7:00 and 11:00 p.m. (or 19:00-23:00 military time) as a possible building time if we want to stay on the beach late in the evening. This interval works because the sine equals -1/2 at $\frac{19\pi}{6}$ and at $\frac{23\pi}{6}$ also. In fact, there are an infinite number of angles for which the sine of the angle is -1/2, but only these four angles make sense in our context of spending a day at the beach. If we turn to the calculator we get $\sin^{-1}\left(-\frac{1}{2}\right) = -0.5236\ or\ -\frac{\pi}{6}$, which really doesn't fit our context at all, since it would mean t = -1, or one hour before midnight of the day we plan to visit the beach. Allow students to wrestle with these ideas until they arrive at similar explanations in terms of the context, as given above. [Wait until the whole class discussion to point out the dilemma that the inverse sine function can only have one output, and it is by convention that mathematicians have restricted output values for the inverse sine function to the interval $\frac{\pi}{2} \le \sin^{-1}(x) \le \frac{\pi}{2}$. Note that the calculator output was in this interval.]

The algebra for question 4 follows the same reasoning as above, however there are no easily identifiable angles on the unit circle for the question, "What angle has a sine value of $^3/_4$?" Consequently, we are left solving this equation using a calculated value for $\sin^{-1}\left(\frac{3}{4}\right) = 0.848$. Solving the equation for t using this value yields t = 1.62, but we can't start building our sand castle

1.62 hours after midnight (approximately 1:37 a.m.), since the level of the tide is rising at that time, not receding. Students will have to reason through how they might use this value to determine the interval of time when they might build their castle. Examining a unit circle diagram would suggest that the sine function also has a value of $^{3}/_{4}$ at an angle equal to π – 0.848 or an angle measuring 2.2936 radians. Solving the equation for *t* using this value yields *t* = 4.38 hours after midnight, or approximately 4:23 a.m. as the time when we can start building our sand castle. To find the time when the rising tide will wash away our sand castle we will need to find the next angle where the sine function equals $^{3}/_{4}$. This occurs at 2π +0.848 or an angle measuring 7.1312 radians. Solving the equation for *t* using this value yields *t* = 13.62 hours after midnight or 1:37 p.m. as the end of the interval in which we can admire our work. (We could also build our sand castle between 4:23 p.m. and 1:37 a.m. the next morning.)

Discuss (Whole Class):
The discussion needs to focus on the ideas and strategies for answering inverse trigonometric questions, as highlighted by the issues addressed above. Discuss the convention for restricting the output of the inverse sine function to the interval $\frac{\pi}{2} \leq \sin^{-1}(x) \leq \frac{\pi}{2}$. This allows the domain of the inverse sine function to cover all possible values of x, $-1 \leq x \leq 1$, and yet yield a unique output. Consequently, the answer obtained using a calculator may not be the answer that fits the context. The unit circle and the graph of the situation serve as aids for figuring out how to interpret calculated values for the inverse sine so that results fit the scenario. Use student work and student thinking about questions 3 and 4 to help make these points. Start with graphs and/or tables to first find the answers to questions 3 and 4, and then pursue the algebraic ideas outlined above.

If there is time, discuss question 6 using tables and graphs to make sense of the scenario. This is not an inverse trigonometric question, since we need only evaluate the given function at particular times to find the lowest point on the beach where we can build the castle. However, it may be difficult to determine the times at which we should evaluate the function without the use of a table or a graph.

Aligned Ready, Set, Go: Trigonometric Functions, Equations and Identities 7.2

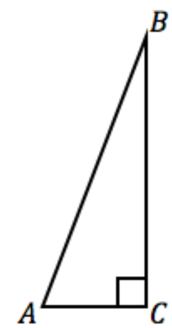
READY, SET, GO! Name Period Date

READY

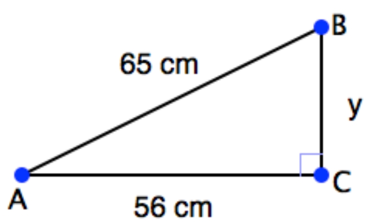
Topic: Calculating tangent in right angle trigonometry

Recall that the right triangle definition of the tangent ratio is:

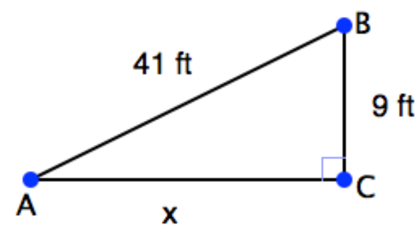
$$\tan A = \frac{\text{length of side opposite angle } A}{\text{length of side adjacent to angle } A}$$



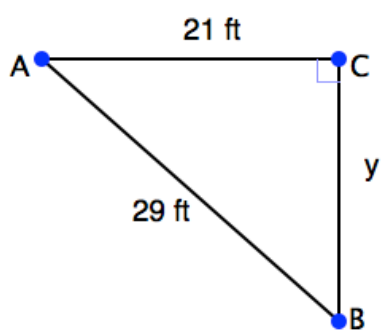
1. Find $\tan A$ *and* $\tan B$.



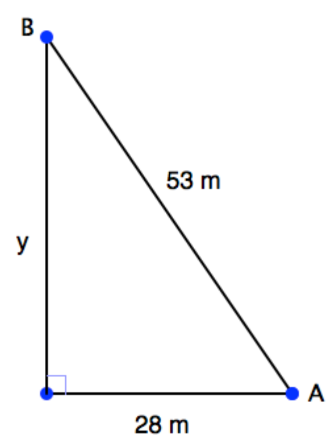
2. Find $\tan A$ *and* $\tan B$.



3. Find $\tan A$ *and* $\tan B$.



4. Find $\tan A$ *and* $\tan B$.



SET

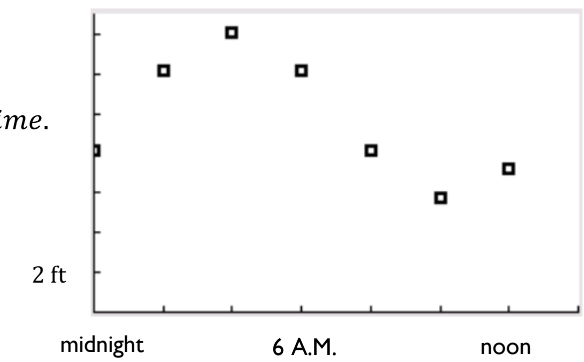
Topic: Mathematical modeling using sine and cosine functions

Many real-life situations such as sound waves, weather patterns, and electrical currents can be modeled by sine and cosine functions. The table below shows the depth of water (in feet) at the end of a wharf as it varies with the tides at various times during the morning.

t (time)	midnight	2 A.M.	4 A.M.	6 A.M.	8 A.M.	10 A.M.	noon
d (depth)	8.16	12.16	14.08	12.16	8.16	5.76	7.26

We can use a trigonometric function to model the data. Suppose you choose cosine.
$y = A\cos(bt - c) + d, where\ y\ is\ depth\ at\ any\ time.$

The *amplitude* will be the distance from the average of the highest and lowest values. This will be the average depth (d).



5. Sketch the line that shows the average depth.

6. Find the amplitude. $A = \frac{1}{2}(high - low)$

7. Find the period. $p = 2|low\ time - high\ time|$. Since a normal period for sine is 2π. The new period for our model will be $\frac{2\pi}{p}\ so\ b = \frac{2\pi}{p}$. (Use the p you calculated, divide and turn it into a decimal.)

8. High tide occurred 4 hours after midnight. The formula for the displacement is $4 = \frac{c}{b}$. Use b and solve for c.

9. Now that you have your values for A, b, c, and d, put them into an equation.
$y = A\cos(bt - c) + d$

10. Use your model to calculate the depth at 9 A.M. and 3 P.M.

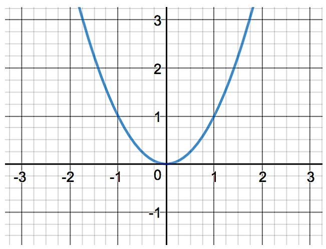
11. A boat needs at least 10 feet of water to dock at the wharf. During what interval of time in the afternoon can it safely dock?

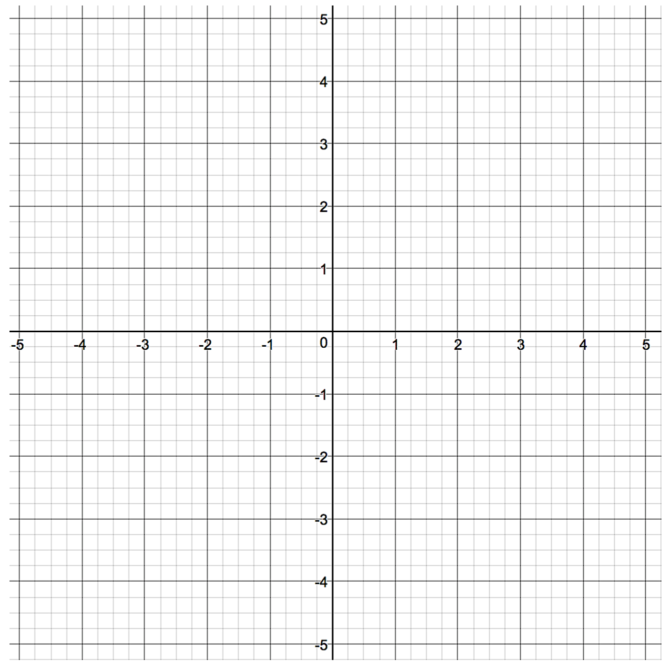
GO

Topic: Connecting transformations on functions

The equation and graph of a parent function is given. For each transformation, describe the change on the graph of the parent function. Then graph the functions on the same grid.

12. $f(x) = x^2$



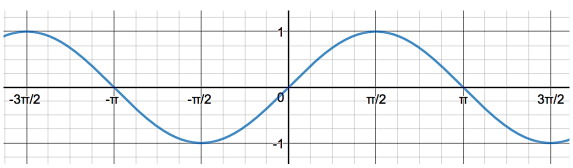


a. $f(x) = x^2 - 3$
   Description:

b. $f(x) = (x-3)^2 - 4$
   Description:

c. $f(x) = 2(x-3)^2 - 4$
   Description:

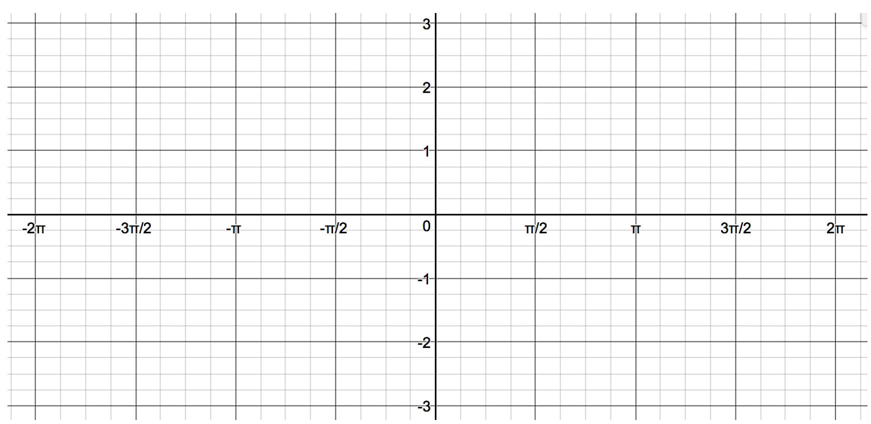
13. $g(x) = sinx$



a. $g(x) = (sinx) + 2$
   Description:

b. $g(x) = sin\left(x + \frac{\pi}{2}\right) - 1$
   Description:

c. $g(x) = 2sin\left(x + \frac{\pi}{2}\right) - 1$
   Description:



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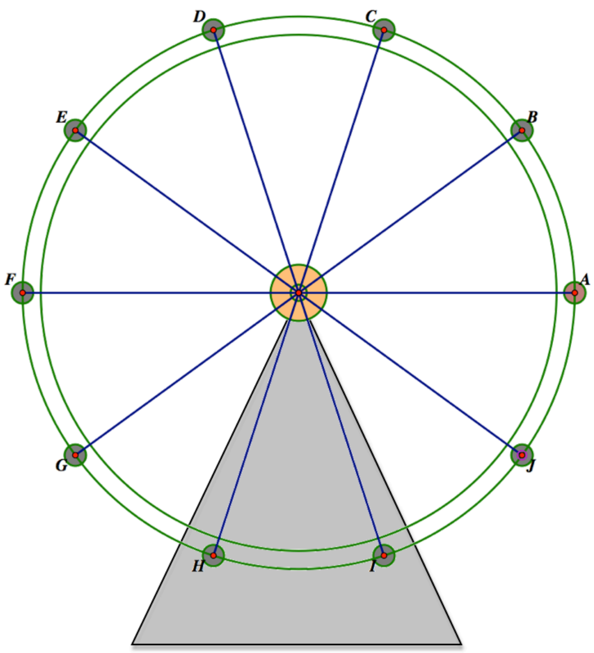
7.3 Getting on the Right Wavelength

A Practice Understanding Task

The Ferris wheel in the following diagram has a radius of 40 feet, its center is 50 feet from the ground, and it makes one revolution counterclockwise every 18 seconds.



1. Write the equation of the height of the rider at any time t, if at $t = 0$ the rider is at position A (Use radians to measure the angle of rotation).

2. At what time(s) is the rider 70 feet above the ground? Show the details of how you answered this question.

3. If you used a sine function in question 1, revise your equation to model the same motion with a cosine function. If you used a cosine function, revise your equation to model the motion with a sine function.

4. Write the equation of the height of the rider at any time t, if at t = 0 the rider is at position D (Use radians to measure the angle of rotation).

5. For the equation you wrote in question 4, at what time(s) is the rider 80 feet above the ground? Show or explain the details of how you answered this question.

6. Choose any other starting position and write the equation of the height of the rider at any time t, if at t = 0 the rider is at the position you chose. (Use radians to measure the angle of rotation). Also change other features of the Ferris wheel, such as the height of the center, the radius, the direction of rotation and/or the length of time for a single rotation. (Record your equation and description of your Ferris wheel here.)

7. Trade the equation you wrote in question 7 with a partner and see if he or she can determine the essential features of your Ferris wheel: height of center, radius, period of revolution, direction of revolution, starting position of the rider. Resolve any issues where you and your partner have differences in your descriptions of the Ferris wheel modeled by your equation.

7.3 Getting on the Right Wavelength – Teacher Notes

A Practice Understanding Task

Purpose: The purpose of this task is to practice the concepts and strategies associated with inverses and transformations of trigonometric functions in a modeling context. Students should use graphs produced with technology to verify their work.

Core Standards Focus:

F.TF.5 Choose trigonometric functions to model periodic phenomena with specified amplitude, frequency, and midline.★

F.BF.3 Identify the effect on the graph of replacing f(x) by f(x) + k, k f(x), f(kx), and f(x + k) for specific values of k (both positive and negative); find the value of k given the graphs. Experiment with cases and illustrate an explanation of the effects on the graph using technology. Include recognizing even and odd functions from their graphs and algebraic expressions for them.

F.BF.4 Find inverse functions.

a. Solve an equation of the form f(x) = c for a simple function f that has an inverse and write an expression for the inverse.

c. (+) Read values of an inverse function from a graph or a table, given that the function has an inverse.

Related Standards: F.BF.1c

Standards for Mathematical Practice:

SMP 2 – Reason abstractly and quantitatively

Vocabulary: For trigonometric functions, a horizontal transformation of a graph is often referred to as a *phase shift*.

The Teaching Cycle:

Launch (Whole Class):

Tell students that they are going to be using all of the trigonometric ideas they have learned in this module and the previous module as they work on this task. Encourage them to use a calculator, set in radian mode, and the unit circle to help them reason through each of the questions. Inform them, however, that their answers to each question should include algebraic work and representations, in addition to answers obtained from a graphical analysis. Students should be able to explain the meaning of each of the parameters in their functions and how they used their functions to answer each question.

Explore (Small Group):

Listen for how students are reasoning through each problem and what tools they draw upon—the context, the Ferris wheel image, the graph, a table, the unit circle, or algebra—to answer each question. For example, once students have written the equation for the height of the rider on the Ferris wheel in question 1, they could use a graph to answer question 2. Acknowledge this approach as appropriate and correct, but also encourage them to think through the algebra of using an inverse sine function and the unit circle to answer the same question. In question 3 students could do some guess and check work to get a cosine graph to match the sine graph from question 1, but press them to explain how they could get the same parameter for the horizontal transformation by reasoning about the period and phase shift of the graph.

Questions 4-5 revisit the same work as questions 1-3, but with a greater level of challenge.

Questions 6-7 provide opportunities for students to create their own scenarios, and may surface additional issues to be resolved or they may highlight students' misconceptions.

Discuss (Whole Class):

Focus the discussion on how students used different tools—the context, the Ferris wheel image, the graph, a table, or algebra—to reason through each question. You may want to have more than one student presentation for each question to highlight different strategies. Connect each way of thinking to the algebraic representations for the problem. For example, on question 2 a student might note that a height of 70 feet is halfway between the center and the top of the Ferris wheel and might also relate this to the angles $\pi/6$ and $5\pi/6$ where the y-coordinate is ½ on the unit circle. Consequently, the rider will have completed $1/12$ (or $5/12$) of a revolution, which will take 1.5 (or 7.5) seconds. This type of reasoning can be related to corresponding work in the algebraic solution:

$$70 = 40\sin\left(\frac{\pi}{9}t\right) + 50$$
$$20 = 40\sin\left(\frac{\pi}{9}t\right)$$
$$\frac{1}{2} = \sin\left(\frac{\pi}{9}t\right)$$
$$\sin^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{9}t$$
$$\frac{\pi}{6} = \frac{\pi}{9}t$$
$$t = \frac{3}{2}\,seconds$$

Reasoning from the diagram or from a graph we can see additional solutions to this question occur at 7. 5 seconds, 19.5 seconds, 25.5 seconds, etc. These times can be listed in the form 1.5 +18n seconds and 7.5 + 18n seconds, where n represents the set of natural numbers (or an appropriate subset, depending on the duration of the ride).

When starting the rider at a different position on the wheel at time t = 0, we need to shift the graph of the height of the rider left or right to correspond with this new starting position. We can reason about this shift in two ways: adding an angle to the angle represented by bt in the equation $h(t) = a\sin(bt + c) + d$, or by adjusting the time when the rider is at a particular starting position by adding a few additional seconds to the time in the equation $h(t) = a\sin(b(t + c)) + d$. If we add to the angle, or to the time, the graph shifts left so that this new starting position occurs at time t = 0.

That is, we are starting at a position that is a few radians or a few seconds into the standard ride. Likewise, if we subtract from the angle, or from the time, the graph shifts right so that this new starting position occurs at time $t = 0$. That is, we are starting at a position that occurred a few radians or a few seconds prior to the standard ride. It is important that both of these ideas come out in the discussion of questions 3 and 4, and that the two transformation expressions $a\sin(bt + c) + d$ and $a\sin(b(t + c)) + d$ are related to each other algebraically. Use student work, if possible, to bring out these two forms, or propose changing one form into the other algebraically and then having students discuss how each form tells the story of the horizontal transformation differently.

End the discussion by connecting equations of the form $y = a\sin(b(x \pm c)) + d$ to a graph, by noting the horizontal transformation represented by the parameter $\pm c$, and relate this shift of the graph to the context to help solidify why the graph moves left or right, based on whether we add or subtract a few seconds represented by the parameter c. Help students notice that when the equation is written in this form the horizontal transformation is easier to identify, and the effect of the parameter added inside the function is consistent with horizontal transformations we have noted in other functions.

Aligned Ready, Set, Go: Trigonometric Functions, Equations and Identities 7.3

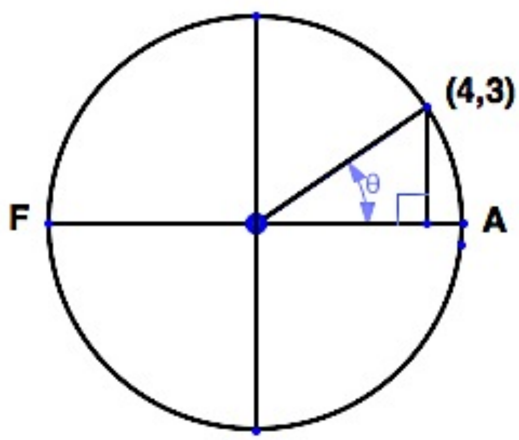
READY, SET, GO! Name Period Date

READY

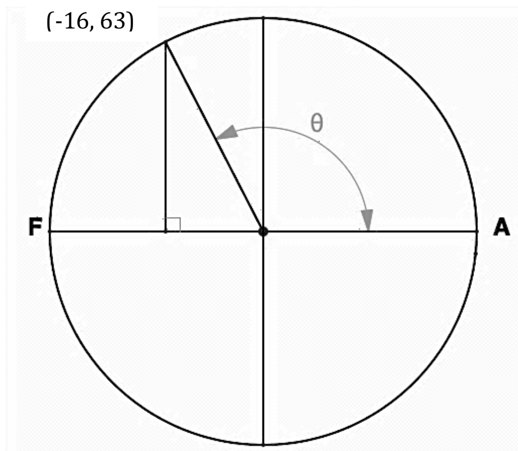
Topic: Using the definition of tangent

Use what you know about the definition of tangent in a right triangle to find the value of tangent θ for each of the right triangles below.

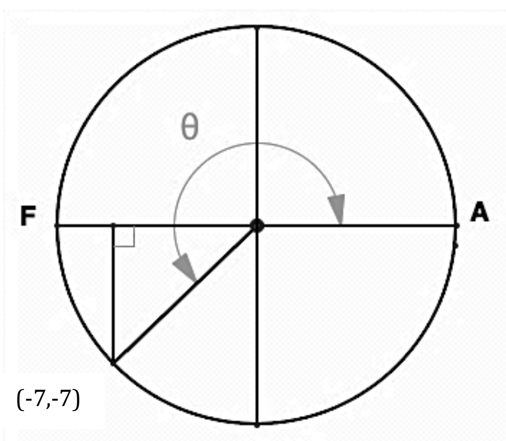
1. $\tan\theta =$



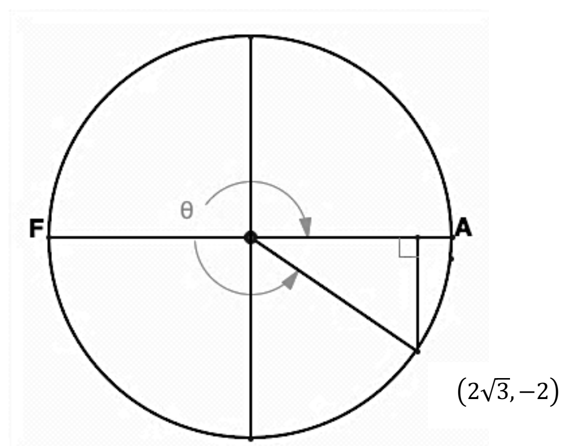
2. $\tan\theta =$



3. $\tan\theta =$



4. $\tan\theta =$



5. In each graph, the angle of rotation is indicated by an arc and θ. Describe the angles of rotation from $0\ to\ 2\pi$ that make tangent θ be positive and the angles of rotation that make tangent θ be negative.

SET

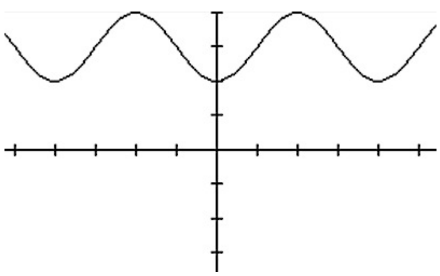
Topic: Connecting trig graphs with their equations

Match each trigonometric representation on the left with an equivalent representation on the right. Then check your answers with a graphing calculator. (The scale on the vertical axis is 1. The scale on the horizontal axis is $\left(\frac{\pi}{2}\right)$.)

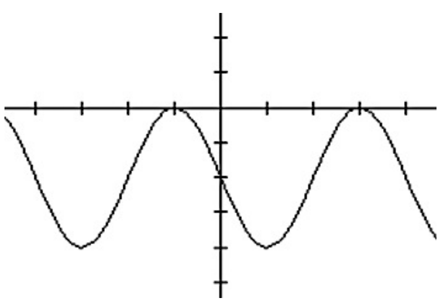
6. $y = -3sin\left(\theta + \frac{\pi}{2}\right)$

7. $y = 3\cos\left(\theta + \frac{\pi}{2}\right)$

8.



9.



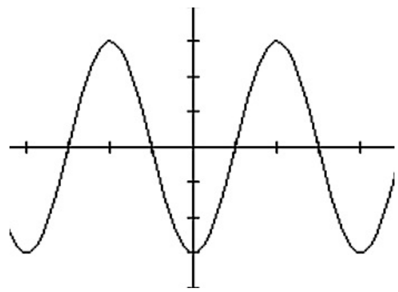
10. $y = \sin\left(2\left(\theta + \frac{\pi}{2}\right)\right) - 2$

11. $y = \sin(x + \pi)$

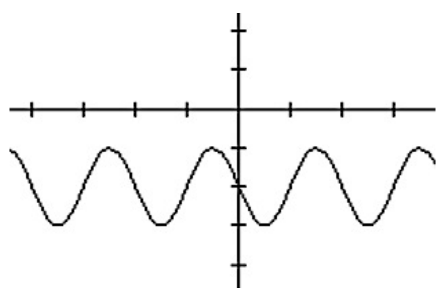
A. $y = -3\sin\theta$

B. $y = -sin\theta$

C.



D.



E. $y = 2cos\left(\theta + \frac{\pi}{2}\right) - 2$

F. $y = \cos(x + \pi) + 3$

12. Choose the equation(s) that has the same graph as $y = cos\theta$.

    a. $y = \cos(-\theta)$

    b. $y = \cos(\theta - \pi)$

    Use the unit circle to explain why they are the same.

13. Choose the equation(s) that has the same graph as $y = -sin\theta$.

    a. $y = \sin(\theta + \pi)$

    b. $y = \sin(\theta - \pi)$

    Use the unit circle to explain why they are the same.

For each function, identify the amplitude, period, horizontal shift, and vertical shift.

14. $f(t) = 150\cos\left(\frac{\pi}{6}(t-8)\right) + 80$

    amplitude:

    period:

    horizontal shift:

    vertical shift:

15. $f(t) = 4.5\sin\left(\frac{\pi}{4}t + \frac{3}{4}\right) + 8$

    amplitude:

    period:

    horizontal shift:

    vertical shift:

GO

Topic: Making sense of composite trig functions

Recall that a composite function places one function such as g(x), inside the other, f(x), by replacing the x in f(x) with the entire function g(x). In general, the notation is $f\left(g(x)\right)$. *Also, recall that inverse functions "undo" each other. Since,* $sin^{-1}\left(\frac{1}{2}\right)$ *is an angle of* $30°$ *because* $sin\,30° = \frac{1}{2}$ *the composite* $sin\left(sin^{-1}\frac{1}{2}\right)$ *is simply* $\frac{1}{2}$. *Sine function "undoes" what* $\sin^{-1}\theta$ *was does.*

Not all composite functions are inverses. Note: problems18 – 24.

Answer the following.

16. $\sin\left(\sin^{-1}\frac{\sqrt{2}}{2}\right)$

17. $\cos\left(\cos^{-1}\frac{\sqrt{3}}{2}\right)$

18. $\tan(\tan^{-1} 9.52)$

19. $\sin\left(\cos^{-1}\frac{1}{2}\right)$

20. $\cos(\tan^{-1} 1)$

21. $\sin(\tan^{-1} 2.75)$

22. $\cos(\sin^{-1} 1)$

23. $\cos(\tan^{-1} 0)$

24. $\sin(\tan^{-1} undefined)$

7.4 Off on a Tangent

A Develop and Solidify Understanding Task

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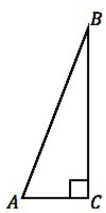
Recall that the right triangle definition of the tangent ratio is:



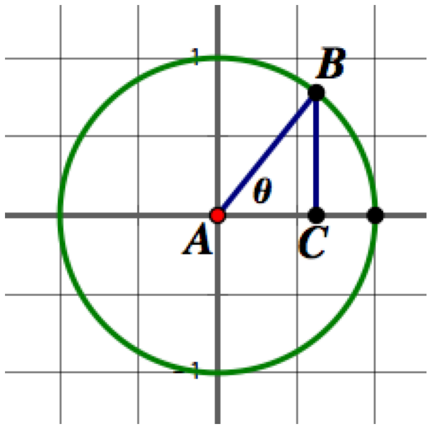
$$\tan(A) = \frac{length\ of\ side\ opposite\ angle\ A}{length\ of\ side\ adjacent\ to\ angle\ A}$$

1. Revise this definition to find the tangent of any angle of rotation, given in either radians or degrees. Explain why your definition is reasonable.

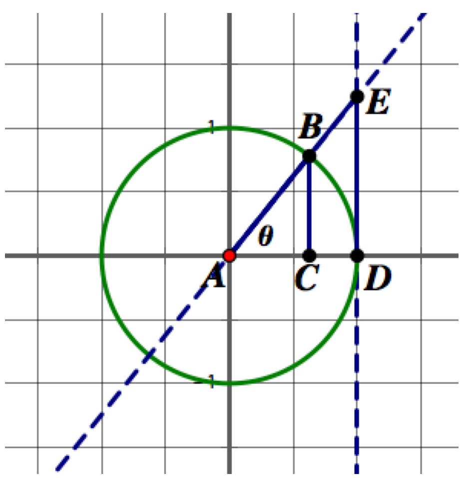
2. Revise this definition to find the tangent of any angle of rotation drawn in standard position on the unit circle. Explain why your definition is reasonable.

We have observed that on the unit circle the value of sine and cosine can be represented with the length of a line segment.

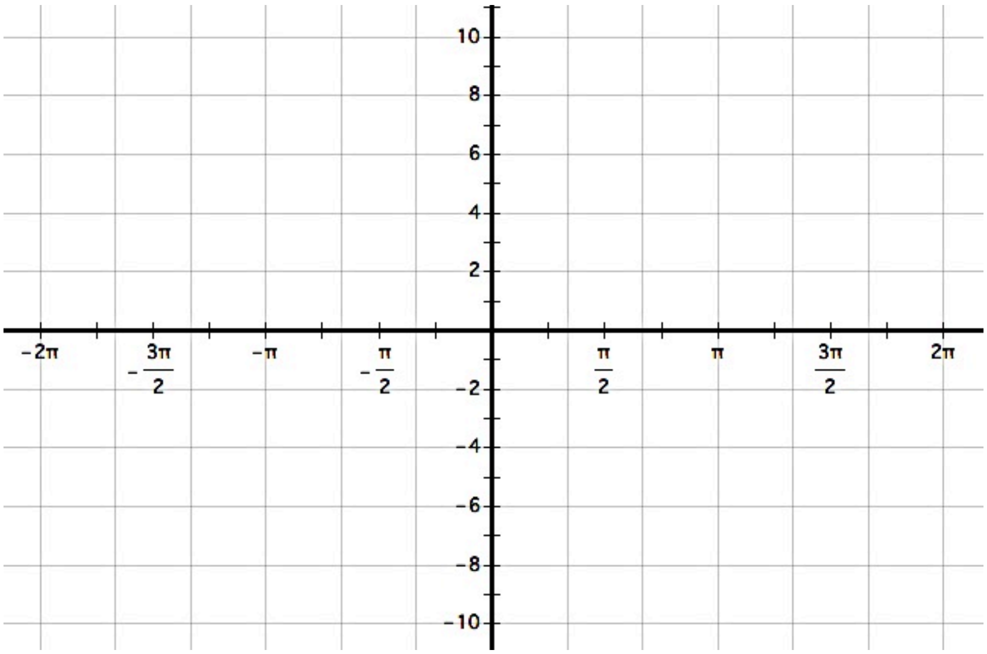
3. Indicate on the following diagram which segment's length represents the value of $\sin(\theta)$ and which represents the value of $\cos(\theta)$ for the given angle θ.



There is also a line segment that can be defined on the unit circle so that its length represents the value of $\tan(\theta)$. Consider the length of $\overline{DE}$ in the unit circle diagram below. Note that ΔADE and ΔABC are right triangles. Write a convincing argument explaining why the length of segment DE is equivalent to the value of $\tan(\theta)$ for the given angle θ.



4. On the coordinate axes below sketch the graph of $y = \tan(\theta)$ by considering the length of segment *DE* as θ rotates through angles from 0 radians to 2π radians. Explain any interesting features you notice in your graph.



Extend your graph of $y = \tan(\theta)$ by considering the length of segment *DE* as θ rotates through negative angles from 0 radians to -2π radians.

5. Using your unit circle diagrams from the task *Water Wheels and the Unit Circle*, give exact values for the following trigonometric expressions:

a. $\tan\left(\frac{\pi}{6}\right) =$

b. $\tan\left(\frac{5\pi}{6}\right) =$

c. $\tan\left(\frac{7\pi}{6}\right) =$

d. $\tan\left(\frac{\pi}{4}\right) =$

e. $\tan\left(\frac{3\pi}{4}\right) =$

f. $\tan\left(\frac{11\pi}{6}\right) =$

g. $\tan\left(\frac{\pi}{2}\right) =$

h. $\tan(\pi) =$

i. $\tan\left(\frac{7\pi}{3}\right) =$

Functions are often classified based on the following definitions:

- A function $f(x)$ is classified as an **odd function** if $f(-\theta) = -f(\theta)$
- A function $f(x)$ is classified as an **even function** if $f(-\theta) = f(\theta)$

6. Based on these definitions and your work in this module, determine how to classify each of the following trigonometric functions.

   - The function $y = sin(x)$ would be classified as an [odd function, even function, neither an odd or even function]. Give evidence for your response.

   - The function $y = cos(x)$ would be classified as an [odd function, even function, neither an odd or even function]. Give evidence for your response.

   - The function $y = tan(x)$ would be classified as an [odd function, even function, neither an odd or even function]. Give evidence for your response.

7.4 Off on a Tangent – Teacher Notes

A Develop and Solidify Understanding Task

Purpose: The purpose of this task is to extend the definition of the tangent from the right triangle trigonometric ratio definition, $\tan(A) = \frac{length\ of\ side\ opposite\ angle\ A}{length\ of\ side\ adjacent\ to\ angle\ A}$, to an angle of rotation definition: $\tan(\theta) = \frac{y}{x}$. The graph of the tangent function is obtained by representing the tangent of an angle of rotation by the length of a line segment related to the unit circle, and tracking the length of the line segment as the angle of rotation increases around the unit circle. The trigonometric identity $\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)}$ is also explored in terms of the unit circle.

Core Standards Focus:
F.TF.2 Explain how the unit circle in the coordinate plane enables the extension of trigonometric functions to all real numbers, interpreted as radian measures of angles traversed counterclockwise around the unit circle.

F.TF.3 (+) Use special triangles to determine geometrically the values of sine, cosine, tangent for **π**/3, **π**/4 and **π**/6, and use the unit circle to express the values of sine, cosine, and tangent for π–x, π+x, and 2π–x in terms of their values for x, where x is any real number.

F.TF.4 (+) Use the unit circle to explain symmetry (odd and even) and periodicity of trigonometric functions.

F.IF.5 Relate the domain of a function to its graph and, where applicable, to the quantitative relationship it describes.

Related Standards: F.IF.4, F.IF.7, F.IF.9

Standards for Mathematical Practice:

SMP 7 – Use appropriate tools strategically

Vocabulary: Students will define the *tangent function* for an angle of rotation as $\tan\theta = \frac{y}{x}$ where x and y are the coordinates of a point on a circle where the terminal ray of the angle of rotation intersects the circle when the angle is drawn in standard position (i.e., the vertex of the angle is at the origin and the initial ray lies along the positive x-axis.)

The Teaching Cycle:

Launch (Whole Class):

Remind students that we have redefined sine and cosine for angles of rotation drawn in standard position by using the values of x, y and r. Ask how they might redefine tangent using these same values. Students should note that the definition $\tan(\theta) = \frac{y}{x}$ is independent of the value of *r.*

Examine the two unit circle drawings in question 3 together as a class. In the drawings label segment AC as $x = \cos(\theta)$, segment BC as $y = \sin(\theta)$ and segment AB as $r = 1$. In the second drawing note that ΔABC is similar to ΔADE and that the measure of segment AE is 1. Using this information ask students to consider what this implies about the measure of segment DE. Give students a couple of minutes to suggest that since the triangles are similar they can write the proportion $\frac{DE}{AE} = \frac{BC}{AC}$ or $\frac{DE}{1} = \frac{y}{x}$. They should recognize that the length of segment DE is defined in the same way that we have defined $\tan(\theta)$. That is, the length of segment DE represents the value of $\tan(\theta)$ in the same way that the length of segment AC represents the value of $\cos(\theta)$ and the length of segment BC represents the value of $\sin(\theta)$. You may also want to point out that the trigonometric identity $\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)}$ is present in this diagram.

Now that we have a way of visually representing the magnitude of the value of $\tan(\theta)$, assign students to work on determining what this implies about the shape and features of the graph of $y = \tan(\theta)$. Also have them work on the rest of the task by using their unit circle diagrams.

Explore (Small Group):

If students are having a hard time sketching the graph, focus their attention on small intervals of θ. For example, what happens to the length of segment DE as θ increases from 0 radians to $\frac{\pi}{2}$ radians? What happens when $\theta = \frac{\pi}{2}$? What happens when θ increases from $\frac{\pi}{2}$ to π? How would you draw ΔADE on this interval? What about negative angles of rotation?

Watch as students compute values of $\tan(\theta)$ using information recorded on their unit circle diagrams. Students may need help simplifying the ratios formed by $\frac{y}{x}$. Allow students to leave these ratios unsimplified until the whole class discussion when you can discuss some of the arithmetic involved, hopefully by using work from students who are successful at simplifying these ratios. Look for such students.

Listen for how students apply the definitions of odd and even functions to the sine, cosine and tangent functions. What representations do they draw upon to make these decisions: the symmetry of points around the unit circle, a graph of the function, or some other ways of reasoning?

Discuss (Whole Class):

Focus the whole class discussion on the following three items:

- The graph of the tangent function, including the period of π and the behavior of the graph near and at $\pm\frac{\pi}{2}$ and $\pm\frac{3\pi}{2}$ (the vertical asymptotes).
- The values of the tangent function at angles that are multiples of $\frac{\pi}{6}$ and $\frac{\pi}{4}$, including the arithmetic of simplifying these ratios.
- The classification of sine, cosine and tangent as even or odd functions and the evidence used to support these classifications (e.g., the graph of the function or the symmetry of the unit circle).

Aligned Ready, Set, Go: Trigonometric Functions, Equations and Identities 7.4

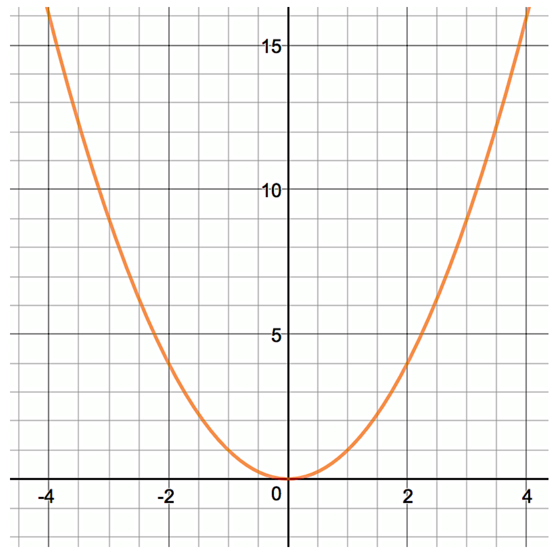
READY, SET, GO! Name Period Date

READY

Topic: Making rigid and non-rigid transformations on functions

The equation of a parent function is given. Write a new equation with the given transformations. Then sketch the new function on the same graph as the parent function. (If the function has asymptotes, sketch them in.)

1. $y = x^2$



Vertical shift: up 8

horizontal shift: left 3

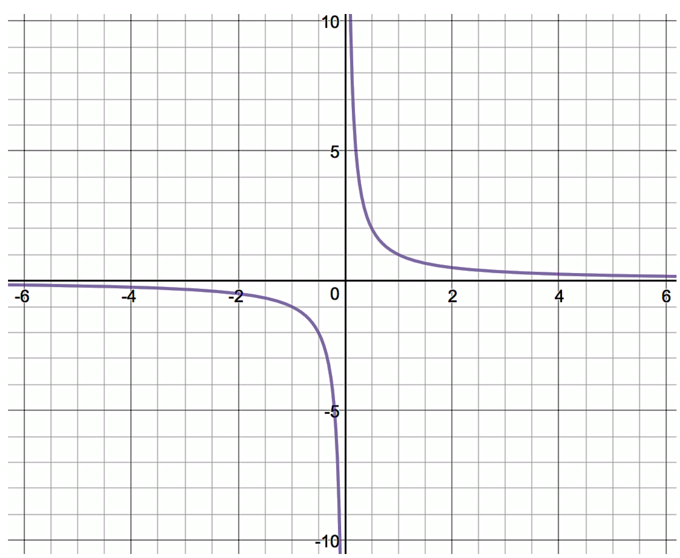
dilation: ¼

Equation:

Domain:

Range:

2. $y = \frac{1}{x}$



Vertical shift: up 4

horizontal shift: right 3

dilation: -1

Equation:

Domain:

Range:

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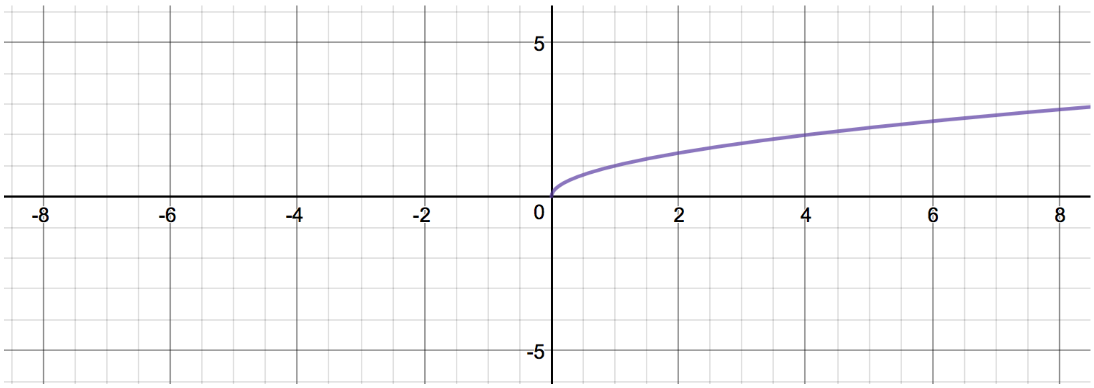
3. $y = \sqrt{x}$

Vertical shift: none. horizontal shift: left 5 dilation: 3

Equation: Domain: Range:



4. $y = \sin x$

Vertical shift: 1

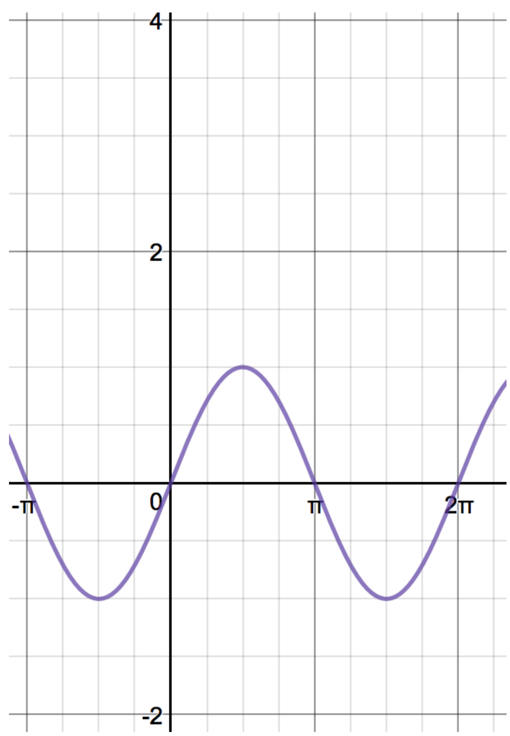
horizontal shift: left $\frac{\pi}{2}$

dilation (amplitude): 3

Equation:

Domain:

Range:

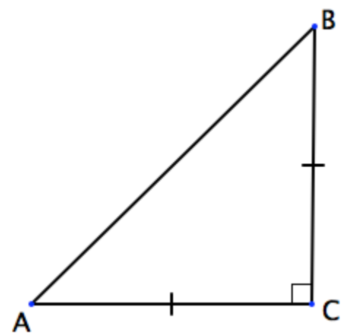


SET

Topic: Connecting values in the special triangles with radian measures

5. Triangle ABC is a right triangle. AB = 1.

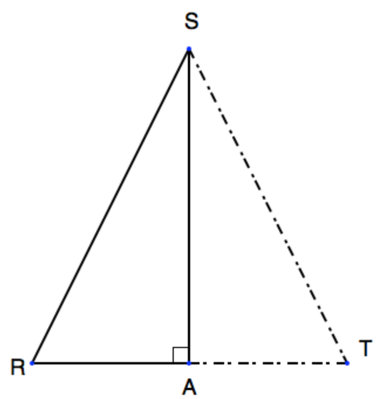
   Use the information in the figure to label the length of the sides and measure of the angles.



6. Triangle RST is an equilateral triangle. RS = 1
   $\overline{SA}$ is an altitude

   Use the information in the figure to label the length of the sides, the length of $\overline{RA}$, and the *exact* length of $\overline{SA}$.

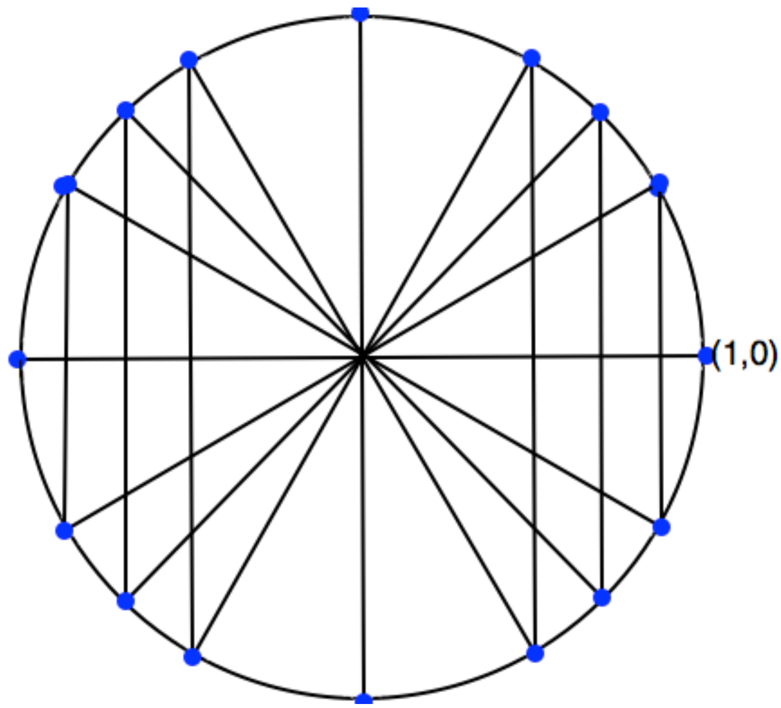
   Label the measure of angles RSA and SRA.



7. Use what you know about the unit circle and the information from the figures in problems 5 and 6 to fill in the table. Some values will be undefined.

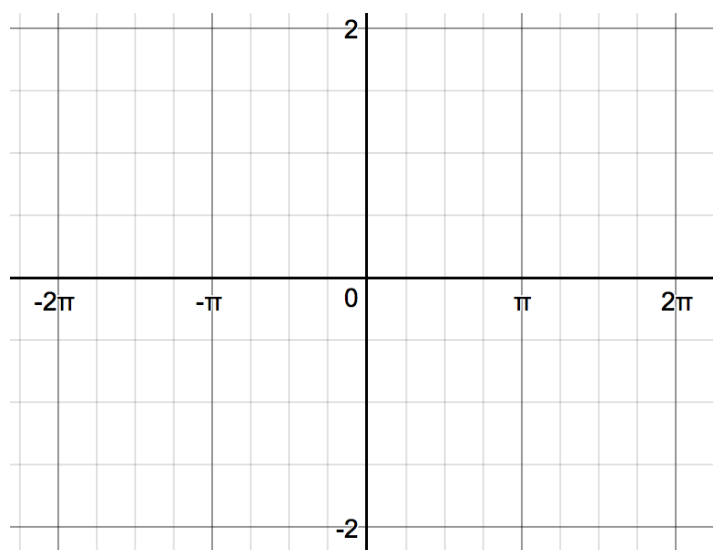
function	$\theta = \frac{\pi}{6}$	$\theta = \frac{\pi}{4}$	$\theta = \frac{\pi}{3}$	$\theta = \frac{\pi}{2}$	$\theta = \pi$	$\theta = \frac{3\pi}{2}$	$\theta = 2\pi$
$\sin\theta$							
$\cos\theta$							
$\tan\theta$							

8. Label all of the points and angles of rotation in the given unit circle.



9. **Graph $f(x) = \tan\theta$.** Use your table of values above for $f(x) = \tan\theta$. Sketch your asymptotes with dotted lines.

10. Where do asymptotes always occur?



GO

Topic: Recalling trig facts

Answer the questions below. Be sure you can justify your thinking.

11. Given triangle ABC with angle C being the right angle, what is the sum of $m\angle A + m\angle B$?

12. Identify the quadrants in which $\sin\theta$ *is positive*.

13. Identify the quadrants in which $\cos\theta$ *is negative*.

14. Identify the quadrants in which $\tan\theta$ *is positive*.

15. Explain why it is impossible for $\sin\theta > 1$.

16. Name the angles of rotation (in radians) for when $\sin\theta = \cos\theta$.

17. For which trig functions do a positive rotation and a negative rotation always give the same value?

18. Explain why in the unit circle $\tan\theta = \frac{y}{x}$.

19. Which function connects with the slope of the hypotenuse in a right triangle?

20. Explain why $\sin\theta = \cos(90° - \theta)$.

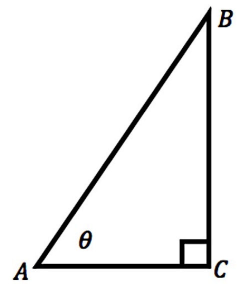
7.5 Maintaining Your Identity

A Develop Understanding Task

Right triangles and the unit circle provide images that can be used to derive, explain and justify a variety of trigonometric identities.

1. For example, how might the right triangle diagram at the right help you justify why the following identity is true for all angles θ between 0° and 90°?

   $$\sin(\theta) = \cos(90° - \theta)$$

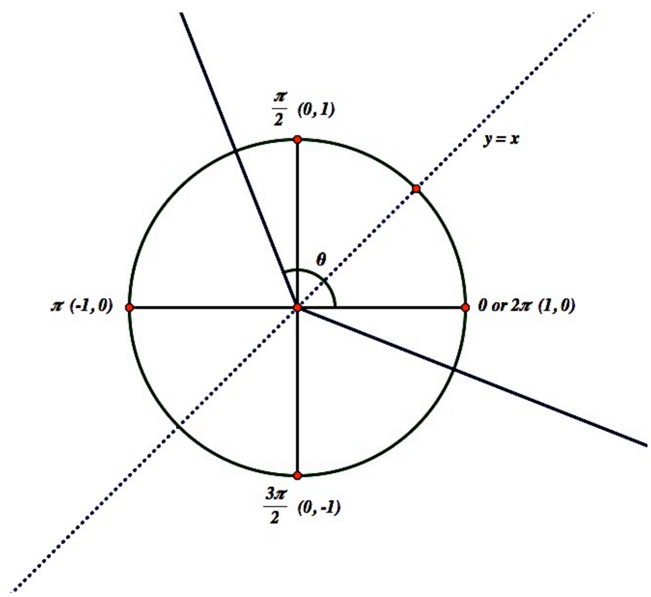


2. Since we have extended our definition of the sine to include angles of rotation, rather than just the acute angles in a right triangle, we might wonder if this identity is true for all angles θ, not just those that measure between 0° and 90°?

   A version of this identity that uses radian rather than degree measure would look like this:

   $$\sin(\theta) = \cos\left(\frac{\pi}{2} - \theta\right)$$

   How might you use this unit circle diagram to justify why this identity is true for all angles θ?



Fundamental Trig Identities

3. Here are some additional trig identities. Use either a right triangle diagram or a unit circle diagram to justify why each is true.

   a. $\sin(-\theta) = -\sin(\theta)$

   b. $\cos(-\theta) = \cos(\theta)$

   c. $\sin^2\theta + \cos^2\theta = 1$ [Note: This is the preferred notation for $(\sin\theta)^2 + (\cos\theta)^2 = 1$]

   d. $\frac{\sin\theta}{\cos\theta} = \tan\theta$

4. Use right triangles or a unit circle to help you form a conjecture for how to complete the following statements as trig identities. How might you use graphs to gain additional supporting evidence that your conjectures are true?

   a. $\sin(\pi - \theta) =$ ____________________ and $\cos(\pi - \theta) =$ ____________________

   b. $\sin(\pi + \theta) =$ ____________________ and $\cos(\pi + \theta) =$ ____________________

   c. $\sin(2\pi - \theta) =$ ____________________ and $\cos(2\pi - \theta) =$ ____________________

5. We can use algebra, along with some fundamental trig identities, to prove other identities. For example, how can you use algebra and the identities listed above to prove the following identities?

   a. $\tan(-\theta) = -\tan(\theta)$

   b. $\tan(\pi + \theta) = \tan(\theta)$

7.5 Maintaining Your Identity – Teacher Notes

A Develop Understanding Task

Purpose: The unit circle and right triangle trigonometry allow students to derive a variety of useful trigonometric identities. The purpose of this task is to practice using diagrams to extend the students' repertoire of identities to include identities related to the odd and even behavior of the sine and cosine functions and the Pythagorean identity.

Core Standards Focus:
F.TF.3 (+) Use the unit circle to express the values of sine, cosine, and tangent for $\pi - x$, $\pi + x$, and $2\pi - x$ in terms of their values for x, where x is any real number.

F.TF.4 (+) Use the unit circle to explain symmetry (odd and even) and periodicity of trigonometric functions.

F.TF.8 Prove the Pythagorean identity $\sin^2(\theta) + \cos^2(\theta) = 1$ and use it to find $\sin(\theta)$, $\cos(\theta)$, or $\tan(\theta)$ given $\sin(\theta)$, $\cos(\theta)$, or $\tan(\theta)$ and the quadrant of the angle.

Standards for Mathematical Practice:
SMP 7 – Look for and make use of structure

Vocabulary: Students will be introduced to the concept of *trigonometric identities*, statements that are true for all values of θ.

The Teaching Cycle:
Launch (Whole Class):
Use the identity on the first page of the task in both its right triangle/degree form and its unit circle/radian form to launch the discussion on using a diagram to justify trig identities. Allow

students a few minutes to individually examine the two diagrams and relate them to the notation in the two versions of the identity. You might ask questions to prompt thinking such as, "Given the marked angle θ, where would the angle 90° – θ, (or the angle $\frac{\pi}{2} - \theta$), appear in the diagram?" Let a couple of students share their reasoning, based on the diagram, as to why the sine of one of these angles is equal to the cosine of the other. For the right triangle diagram students should notice that the ratio of sides defining the sine of one of the angles is the same ratio of sides that defines the cosine of the other angle. For the unit circle diagram they should note that the terminal ray of the angle that starts at 0 and rotates counterclockwise θ radians and the terminal ray of the angle that starts at $\pi/2$ and rotates clockwise θ radians are mirror images about the line $y = x$. Therefore, the x and y coordinates of the points where these terminal rays intersect the unit circle are switched. Since these coordinates define the magnitudes of the sine and cosine of the angles on the unit circle, the sine of one of the angles is the same as the cosine of the other.

Inform students that they are to continue to use right triangles or unit circle diagrams to explore, illustrate and justify each of the statements listed in the remainder of the task.

Explore (Small Group):

Questions 3a and 3b can best be illustrated and justified on a unit circle diagram, since the identity requires students to think about the relationship between angles drawn in both positive and negative directions, which would not make sense in a right triangle.

Question 3c can be justified on a right triangle using the Pythagorean theorem, or on the unit circle using the equation of a circle of radius 1.

Question 3d can be justified using ratios of sides on a right triangle, or using the angle of rotation definitions of sine, cosine and tangent based on the x and y coordinates of points on the unit circle.

Note that questions 4a-4d ask students to conjecture an identity by completing the equality statements, and that a graph can be used as additional support for exploring or justifying students' conjectures. If students are having difficulty conjecturing how to complete one of the identities, suggest that they graph the given expression as a function. That is, graphing $y = \sin(\pi - x)$ can lead students to conjecture that $\sin(\pi - x) = \sin(x)$.

Questions 5a and 5b ask students to make an algebraic argument, rather than a geometric one. These problems can be used to extend the work of the task for some students, while allowing time for other students to finish questions 1-4. When a few students have finished questions 5a and 5b, or if students are struggling with the algebra, move to the whole class discussion.

Discuss (Whole Class):

Select students to present their justifications of the identities in questions 1-3 and their conjectures and justifications for the identities in questions 4. If there are justifications that are based on both a right triangle diagram and a unit circle diagram you might want to have both justifications presented in order to reinforce that these identities are true—where applicable—using both the "right triangle ratio of sides" trig definition and the "coordinates of points on a unit circle" trig definition of sine and cosine.

Conclude the discussion by having students present their algebraic justifications of the two statements in question 5, or work through the algebra of statement 5a together as a class and then give students time to work through statement 5b on their own. The essence of proving these statements about tangent functions is to use the identity given in statement 3d to turn the tangent into the ratio of the sine and cosine, and then apply identities we have written for the sine and cosine. This algebra work will give students a reason for creating a collection of identities in the first place, and will prepare students for some of the algebraic work that will occur in the following task.

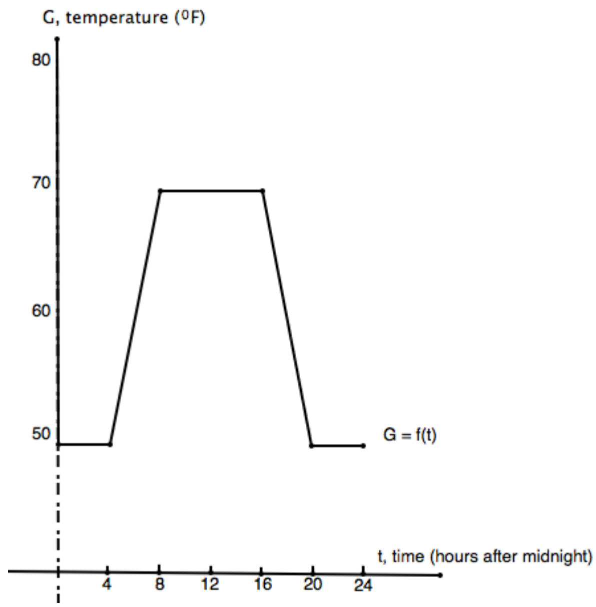
Aligned Ready, Set, Go: Trigonometric Functions, Equations and Identities 7.5

READY, SET, GO! Name Period Date

READY

Topic: Modeling transformations

A school building is kept warm only during school hours, in order to save money. Figure 6.1 shows a graph of the temperature, G, in °F, as a function of time, t, in hours after midnight. At midnight (t = 0), the building's temperature is 50 °F. This temperature remains the same until 4 AM. Then the heater begins to warm the building so that by 8 am the temperature is 70°F. That temperature is maintained until 4 pm, when the building begins to cool. By 8 pm, the temperature has returned to 50°F and will remain at that temperature until 4 am.



1. In January many students are sick with the flu. The custodian decides to keep the building 5 °F warmer. Sketch the graph of the new schedule on figure 6.1.

2. If $G = f(t)$ is the function that describes the original temperature setting, what would be the function for the January setting?

3. In the spring, the drill team begins early morning practice. The custodian then changes the original setting to start 2 hours earlier The building now begins to warm at 2 am instead of 4 am and reaches 70 °F at 6 am. It begins cooling off at 2 pm instead of 4 pm and returns to 50 °F at 6 pm instead of 8 pm. Sketch the graph of the new schedule on figure 6.1.

4. If $G = f(t)$ is the function that describes the original temperature setting, what would be the function for the spring setting?

SET

Topic: Using trigonometric identities to find additional trig values

The *Cofunction identity* states: $\sin\theta = \cos\left(\frac{\pi}{2} - \theta\right)$ *and* $\cos\theta = \sin\left(\frac{\pi}{2} - \theta\right)$

Complete the statements, using the *Cofunction identity.*

5. $\text{Sin } 70° = \cos$ ______°
6. $\text{Sin } 28° = \cos$ ______°
7. $\text{Cos } 54° = \sin$ ______°
8. $\text{Sin } 9° = \cos$ ______°
9. $\text{Cos } 72° = \sin$ ______°
10. $\text{Cos } 45° = \sin$ ______°
11. $\text{Cos } \frac{\pi}{8} = \sin$ ——
12. $\text{Sin } \frac{5\pi}{12} = \cos$ ——
13. $\text{Sin } \frac{3\pi}{10} = \cos$ ——

14. Let $\sin\theta = \frac{3}{4}$.

    a) Use the *Pythagorean identity* $(sin^2\theta + cos^2\theta = 1)$, to find the value of $\cos\theta$.

    b) Use the *Quotient identity* $\left(\tan\theta = \frac{\sin\theta}{\cos\theta}\right)$, the given information, and your answer in part (a) to calculate the value of $\tan\theta$.

15. Let $\cos\beta = \frac{12}{13}$.

    a) Find $\sin\beta$. Use the *Pythagorean identity* $(sin^2\theta + cos^2\theta = 1)$.

    b) Find $\tan\beta$. Use the *Quotient identity* $\left(\tan\theta = \frac{\sin\theta}{\cos\theta}\right)$.

    c) Find $\cos\left(\frac{\pi}{2} - \theta\right)$. Use a *Cofunction identity.*

Use trigonometric definitions and identities to prove the statements below.

16. $\tan\theta\cos\theta = \sin\theta$

17. $(1+\cos\beta)(1-\cos\beta) = \sin^2\beta$

18. $(1+\sin\alpha)(1-\sin\alpha) = \cos^2\alpha$

19. $sin^2W - cos^2W = 2sin^2W - 1$

GO

Topic: Solving simple trig equations using the special angle relationships

Find __two__ solutions of the equation. Give your answers in degrees $(0° \leq \theta \leq 360°)$ and radians $(0 \leq \theta \leq 2\pi)$. Do NOT use a calculator.

20. $\sin\theta = \frac{1}{2}$

degrees: ______________________________

radians: ______________________________

21. $\sin\theta = -\frac{1}{2}$

degrees: ______________________________

radians: ______________________________

22. $\cos\theta = \frac{\sqrt{2}}{2}$

degrees: ______________________________

radians: ______________________________

23. $\sin\theta = -\frac{\sqrt{3}}{2}$

degrees: ______________________________

radians: ______________________________

24. $\tan\theta = -1$

degrees: ______________________________

radians: ______________________________

25. $\tan\theta = \sqrt{3}$

degrees: ______________________________

radians: ______________________________

7.6 Hidden Identities

A Practice Understanding Task

Note: Because trig functions are periodic, trig equations often have multiple solutions. Typically, we are only interested in the solutions that lie within a restricted interval, usually the interval from 0 to 2π. In this task you should find all solutions to the trig equations that occur on $[0, 2\pi]$.

To sharpen their trig skills, Alyce, Javier and Veronica are trying to learn how to solve some trig equations in a math refresher text that they found in an old trunk one of the adults had brought to the archeological site. Here is how each of them thought about one of the problems:

$$\textit{Solve}: \cos\left(\frac{\pi}{2} - \theta\right) = \frac{1}{2}$$

Alyce: I used the inverse cosine function.

Javier: I first used an identity, and then an inverse trig function. But it was not the same inverse trig function that Alyce used.

Veronica: I graphed $y_1 = \cos\left(\frac{\pi}{2} - \theta\right)$ and $y_2 = \frac{1}{2}$ on my calculator. I seem to have found more solutions.

1. Using their statements as clues, go back and solve the equation the way that each of the friends did.

2. How does Veronica's solutions match with Alyce and Javier's? What might be different?

Solve each of the following trig equations by adapting Alyce and Javier's strategies: that is, you may want to see if the equation can be simplified using one of the trig identities you learned in the previous task; and once you have isolated a trig function on one side of the equation, you can undo that trig function by taking the inverse trig function on both sides of the equation. Once you have a solution, you may want to check to see if you have found all possible solutions on the interval $[0, 2\pi]$ by using a graph as shown in Veronica's strategy.

3. $\sin(-\theta) = -\frac{1}{2}$

4. $\cos\theta \cdot \tan\theta = \frac{\sqrt{3}}{2}$

5. $\sin(2\theta) = \frac{\sqrt{2}}{2}$

7.6 Hidden Identities – Teacher Notes

A Practice Understanding Task

Purpose: In this task, students draw upon some of the identities discovered in the previous task to help them change the form of trigonometric expressions into simpler expressions until the solutions of the equation can be found. The dilemma of finding all of the solutions to an equation when using an inverse trig function to solve a trig equation surfaces again, and students have an opportunity to again consider the meaning of solutions to an equation.

Core Standards Focus:

F.TF.7 (+) Use inverse functions to solve trigonometric equations that arise in modeling contexts; evaluate the solutions using technology, and interpret them in terms of the context.*

Standards for Mathematical Practice:

SMP 7 – Look for and make use of structure

Vocabulary: Students will look for all solutions to each trigonometric equation within *restricted intervals* of the variable, such as solving for θ on the interval $[0, 2\pi]$.

The Teaching Cycle:

Launch (Whole Class):

Introduce the equation that Alyce, Javier and Veronica are trying to solve, $\cos\left(\frac{\pi}{2} - \theta\right) = \frac{1}{2}$, and then have students try out each of the three strategies described in the task for solving this equation. Make sure that students are using each of these three uniquely different strategies, and not just solving the problem using a single strategy. After a few minutes, have three students demonstrate each of the three strategies, including Veronica's graphical method, which reveals that this equation has an infinite number of solutions. Have students explain how they could get all of Veronica's solutions from the solutions obtained algebraically using Alyce and Javier's approaches. Then read the statement at the beginning of the task about restricting the solutions to those that occur on the interval $[0, 2\pi]$. Remind students that they should find all solutions within this interval for each trig equation. Students should now be ready to work on problems 3-5 by using a symbolic strategy similar to the work of either Alyce or Javier.

Explore (Small Group):

Encourage students to try to find all solutions to each equation on the interval $[0, 2\pi]$ before they check their algebraic work by using a graph. If the graph reveals more solutions than obtained by the students through their symbolic work, ask them to explain why they might be missing solutions, and how they might look for those solutions without using a graph. This may be particularly necessary on question 5, since two sets of solutions occur on the interval $[0, 2\pi]$ since the period of the trigonometric expression on the left side of the equation has been compressed. Question 5 may create other issues, since the argument of the trig expression is 2θ, instead of just θ.

Discuss (Whole Class):

Discuss each of the three problems thoroughly, listening for misconceptions in students work. For example, in question 3, why can we eliminate the negative signs that appear on both sides of the equation? Don't allow students to just say, "I cancelled out the negative signs on both sides of the equation" or "I got rid of the negative signs by multiplying both sides of the equation by -1." Students need to recognize that the negative sign inside the sine function plays a different role than the negative sign on the fraction on the right side of the equation. They should describe their first step in solving this equation as replacing $\sin(-\theta)$ with $-\sin(\theta)$ based on the identity $\sin(-\theta) = -\sin(\theta)$, and then multiplying both sides of the equation by -1. For question 4, students should use the identity $\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)}$ and then recognize that $\frac{\cos(\theta)}{\cos(\theta)} = 1$. On question 5 students will need to find all angles of rotation where the sine value is $\frac{\sqrt{2}}{2}$ during two revolutions counterclockwise around the unit circle, since they will be dividing these values by 2 to solve for θ on the interval $[0, 2\pi]$.

Aligned Ready, Set, Go: Trigonometric Functions, Equations and Identities 7.6

READY, SET, GO! Name Period Date

READY

Topic: Using the calculator to find angles of rotation

Use your calculator and what you know about where sine and cosine are positive and negative in the unit circle to find the <u>two angles</u> that are solutions to the equation. Make sure $\boldsymbol{\theta}$ *is always* $\mathbf{0} < \boldsymbol{\theta} \leq \mathbf{2}\boldsymbol{\pi}$. Round your answers to 4 decimals.
(Your calculator should be set in radians.)

You will notice that your calculator will sometimes give you a negative angle. That is because the calculator is programmed to restrict the angle of rotation so that the inverse of the function is also a function. Since the requested answers have been restricted to positive rotations, if the calculator gives you a negative angle of rotation, you will need to figure out the positive coterminal angle for the angle that your calculator has given you.

1. $\sin\theta = \frac{4}{5}$
2. $\sin\theta = \frac{2}{7}$
3. $\sin\theta = -\frac{1}{10}$
4. $\sin\theta = -\frac{13}{14}$

5. $\cos\theta = \frac{11}{12}$
6. $\cos\theta = \frac{1}{9}$
7. $\cos\theta = -\frac{7}{8}$
8. $\cos\theta = -\frac{2}{5}$

*Note: When you ask your calculator for the angle, you are "undoing" the trig function. Finding the angle is finding the inverse trig function. When you see "***Find** $\mathbf{\sin^{-1}\left(\frac{4}{5}\right)}$*" , you are being asked to find the angle that makes it true. The answer would be the same as the answer your calculator gave you in #1. Another notation that means the inverse sine function is* $\mathbf{arcsin}\left(\frac{4}{5}\right)$.

SET

Topic: Verifying trig identities with tables, unit circles, and graphs

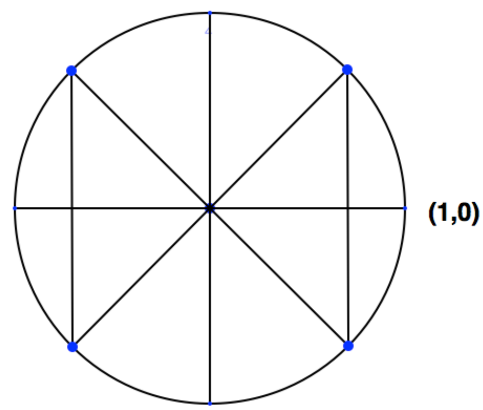
9. **Use the values in the table to verify the Pythagorean identity** $\left(\boldsymbol{sin^2\theta + cos^2\theta = 1}\right)$**.** **Then use the Quotient Identity** $\left(\boldsymbol{tan\,\theta = \frac{sin\theta}{cos\theta}}\right)$ **and the values in the table to write the value of tangent** $\boldsymbol{\theta}$**.**

9.	$\sin\theta$	$\cos\theta$	$\sin^2\theta + \cos^2\theta = 1$	$\tan\theta$
a.	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$		
b.	$\frac{-3}{5}$	$\frac{4}{5}$		
c.	$-\frac{5}{13}$	$-\frac{12}{13}$		
d.	$\frac{\sqrt{14}}{7}$	$\frac{\sqrt{35}}{7}$		
e.	-1	0		
f.	0	1		

10. Label the angles of rotation and the coordinate points around the unit circle on the right. Then use these points to help you fill in the blank.

$\cos(\pi - \theta) =$ ____________________

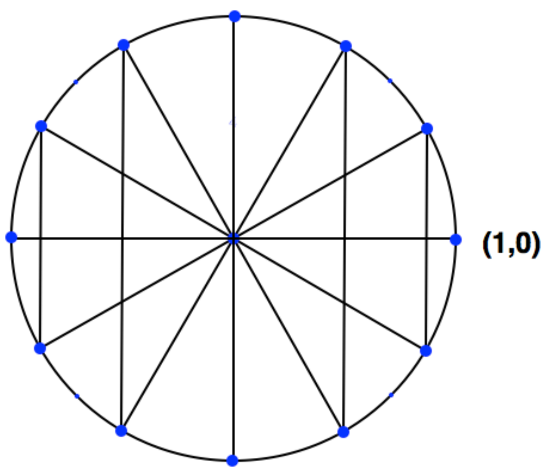
Make your thinking visible by using the diagram. Explain your reasoning.



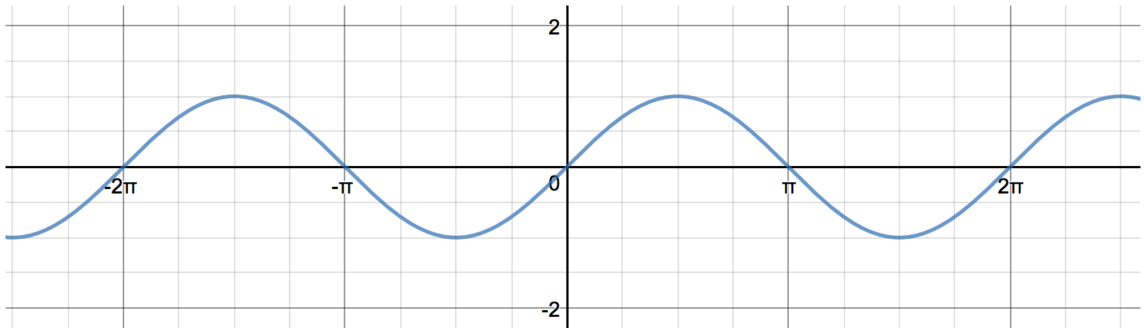
11. Label the angles of rotation and the coordinate points around the unit circle on the right. Then use these points to help you fill in the blank.

$\sin(\pi + \theta) =$ ____________________

Make your thinking visible by using the diagram. Explain your reasoning.



12. Use the graph of $\sin\theta$ to help you fill in the blank. $sin(2\pi - \theta) =$ ________________________



Make your thinking visible by using the graph. Explain your reasoning.

GO

Topic: Finding the central angle when given arc length and radius

Find the radian measure of the central angle of a circle of radius *r* that intercepts an arc of length *s*. $(\boldsymbol{s = r\theta})$

Round answers to 4 decimal places.

Radius	*Arc Length*	*Angle measure in radians*
13. 35 mm	11 mm	
14. 14 feet	9 feet	
15. 16.5 m	28 m	
16. 45 miles	90 miles	